Mth 146: Day 21

10.1 Curves Defined by Parametric Equations
10.2 Calculus with Parametric Curves

1 Part 1: New Concepts & Worked Examples

10.1 Curves Defined by Parametric Equations

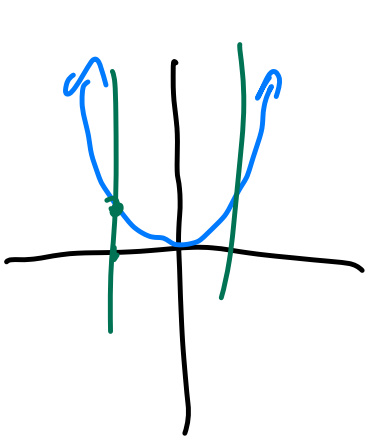
Some curves fail the Vertical Line Test, so we can't write them as $y = f(x)$. Instead, we describe both x and y in terms of a third variable.

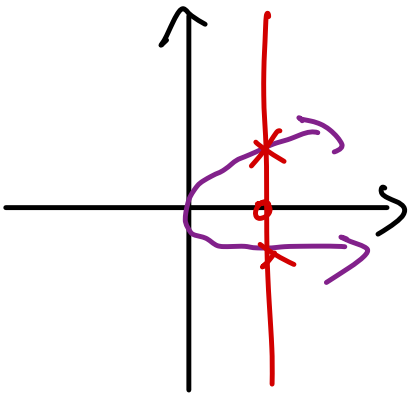
Definition. If x and y are both functions of a third variable t, called the parameter,

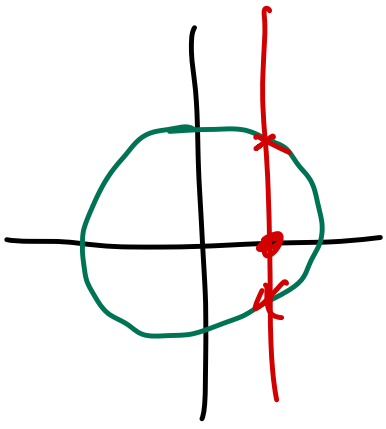
$$x = f(t) \qquad y = g(t)$$

these are called parametric equations. As t varies, the point $(x, y) = (f(t), g(t))$ traces out a parametric curve.

Think of t as **time**: $(f(t), g(t))$ is the position of a moving particle at time t.







yes

Example 1. Sketch and identify the curve

$x = t^2 - 2t, \quad y = t + 1.$

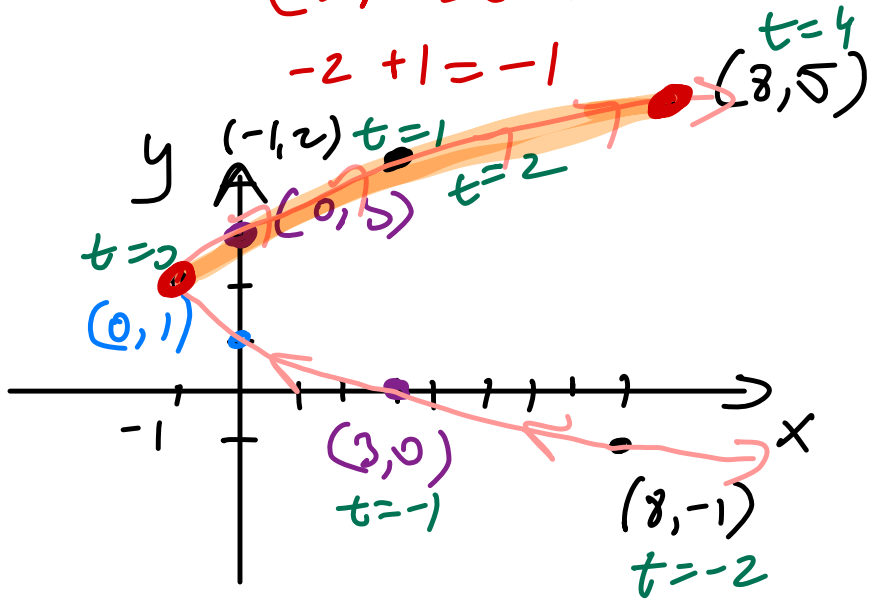
$x = f(t)$ $\quad y = g(t)$

$-2 \le t \le 4$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

$(-2)^2 - 2(-2) = 4 + 4 = 8$

$-2 + 1 = -1$



Eliminate the parameter t:

$y = t + 1 \rightarrow t = y - 1$

Now $x = t^2 - 2t$ becomes $x = (y-1)^2 - 2(y-1)$

$x = y^2 - 4y + 3$ parabola opening to right

Eliminating the parameter (finding an equation in only x and y) helps identify the shape, but we lose direction and timing of motion.

If we restrict $a \le t \le b$, the curve has **initial point** $(x_0, y_0) = (f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

Example: In Example 1 with $0 \le t \le 4$, the curve starts at $(0,1)$ and ends at $(8,5)$.

Example 2. What curve is given by $x = \sin 2t,\ y = \cos 2t,\ 0 \le t \le 2\pi$?

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Note: Different parametric equations can trace the _____ in different ways (direction, speed, number of times around).

Example 3. Consider the three parametrizations

(a) $x = \sin t, \quad y = \cos t, \quad 0 \le t \le \pi$

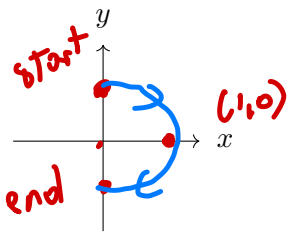
(b) $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

(c) $x = \sin 2t, \quad y = \cos 2t, \quad 0 \le t \le 2\pi$

All three trace portions of the same curve.

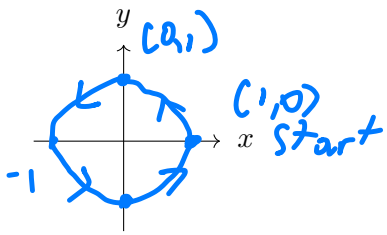
- What is the curve?
- What is the initial position and the direction traced by each one?
- How many times is it traced?

(a)



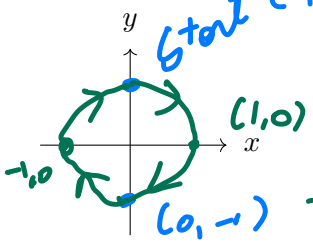
t	$x(t)$	$y(t)$
0	0	1
$\pi/2$	1	0
π	0	−1

(b)



t	$x(t)$	$y(t)$
0	1	0
$\pi/2$	0	1
π	−1	0
$3\pi/2$	0	−1
2π	1	0

(c)



t	$x(t)$	$y(t)$
0	0	1
$\pi/4$	1	0
$\pi/2$	0	−1
$3\pi/4$	−1	0
π	0	1

Curve: _____

(a) Start: _____ Direction: _____ Times: _____

(b) Start: _____ Direction: _____ Times: _____

(c) Start: _____ Direction: _____ Times: _____

Circle with center (h, k) and radius r:

$x = h + r\cos t$ $\quad y = k + r\sin t$ $\quad 0 \le t \le 2\pi$

(traced once, counterclockwise)

Example 4. Sketch the curve $x = \sin t$, $y = \sin^2 t$.

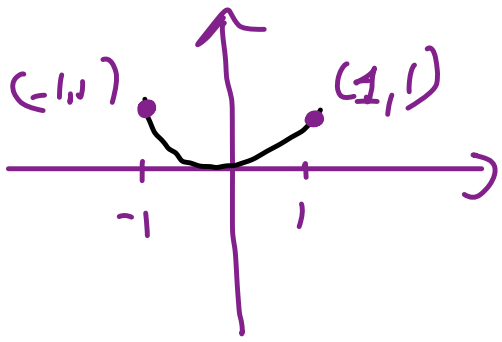
Eliminate parameter:

$y = \sin^2 t$

$y = x^2$

$-1 \le \sin t \le 1$

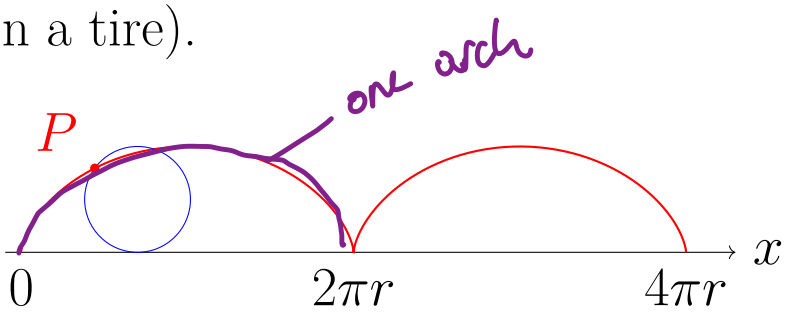
$-1 \le x \le 1$



Graphing tip: Any curve $x = g(y)$ can be graphed on a calculator using $x = g(t)$, $y = t$. Likewise, $y = f(x)$ is the parametric curve $x = t$, $y = f(t)$.

The Cycloid

A cycloid is the curve traced by a point P on the rim of a circle of radius r as the circle rolls along the x-axis (like a pebble stuck in a tire).



Using the angle of rotation θ as the parameter, the cycloid is

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

One arch comes from one full turn of the circle: $0 \le \theta \le 2\pi$.

10.2 Calculus with Parametric Curves

Tangents

Slope of the tangent line. If $x = f(t)$, $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if } \frac{dx}{dt} \neq 0.$$

- **Horizontal tangent** when $dy/dt = 0$ (and $dx/dt \neq 0$).
- **Vertical tangent** when $dx/dt = 0$ (and $dy/dt \neq 0$).

Second derivative. Replace y by $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

Careful: $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2y/dt^2}{d^2x/dt^2}$.

Example 5. A curve C is defined by $x = t^2$, $y = t^3 - 3t$.

(a) Show that C has two tangents at $(3, 0)$ and find their equations.

(b) Find the points where the tangent is horizontal or vertical.

(c) Determine where the curve is concave up or down.

(d) Sketch the curve.

a) need to find t when particle reaches $(3,0)$ (x, y)

$x = t^2$

$3 = t^2 \rightarrow t = \pm\sqrt{3}$

Example 6. (a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at $\theta = \pi/3$.
(b) Where is the tangent horizontal?

slope when $t = -\sqrt{3}$ $\quad \frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = -\sqrt{3}$

second tangent line $\quad y = -\sqrt{3}(x-3)$

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find tangents @ $(3,0)$ $\longrightarrow$ $t = \pm\sqrt{3}$

$y - y_0 = m(x - x_0)$

$y - 0 = m(x-3)$ need slope

slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$

slope for $t = \sqrt{3}$ $\quad \frac{3(\sqrt{3})^2-3}{2(\sqrt{3})} = \frac{3\cdot 3-3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$

eqn of tangent $\quad y = \sqrt{3}(x-3)$