Mth 146: Day 7
7.5 Strategy for Integration
(Solutions)

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1 Part 1: Identify the Strategy

Over the last several days, we've built up a whole toolbox of integration techniques: u-substitution, integration by parts, trig integrals, trig substitution, and partial fractions. Today we throw them all together! The integrals below are presented in random order, with no hints about which method applies. Your job is to play detective — look closely at the structure of each integrand and figure out which strategy fits. Don't worry about getting it wrong on the first guess; the real skill we're building is recognizing the pattern, and the only way to get good at that is to try, and try again.

Strategy for Integration:

1. Simplify the integrand if possible, making sure the rational expression is **proper**.
2. Look for an obvious **substitution**.
3. If the integrand is a product involving a polynomial, trig, exponential, or logarithmic function, try **integration by parts**.
4. If $f(x)$ is a product of powers of $\sin x$ and $\cos x$, $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$, then use the rules for **trig integration**.
5. If $f(x)$ is a rational function that's not obvious how to integrate, try **partial fractions**.
6. If $\sqrt{\pm a^2x^2 \pm b^2}$ occurs, try **trig substitution**.
7. If $\sqrt[n]{g(x)}$ occurs, rationalize it by using a u**-sub**.
8. Sometimes you need to manipulate the integrand and use several methods. Just try and try.

For each of the following integrals, determine a strategy for evaluating. Don't evaluate them, just figure out which technique of integration will work, including what substitution you would use.

1. $\displaystyle\int x\sqrt{9-x^2}\,dx$

   Strategy: u-sub, $u = 9 - x^2$.

   Answer: $-\dfrac{1}{3}(9-x^2)^{3/2} + C$.

2. $\displaystyle\int \frac{3}{x^2+5x+4}\,dx$

   Strategy: partial fractions, $(x+4)(x+1)$.

   Answer: $\ln\left|\dfrac{x+1}{x+4}\right| + C$.

3. $\displaystyle\int \arcsin x\,dx$

   Strategy: integration by parts, $u = \arcsin x$, $dv = dx$.

   Answer: $x\arcsin x + \sqrt{1-x^2} + C$.

4. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$

   Strategy: by parts, $u = \ln(x+1)$, $dv = dx/x^2$; leftover is partial fractions.

   Answer: $-\dfrac{\ln(x+1)}{x} + \ln\left|\dfrac{x}{x+1}\right| + C$.

5. $\displaystyle\int \frac{e^x}{4+e^{2x}}\,dx$

   Strategy: u-sub, $u = e^x$ (arctangent form).

   Answer: $\dfrac{1}{2}\arctan\left(\dfrac{e^x}{2}\right) + C$.

6. $\displaystyle\int \frac{1}{\sqrt{x}(2+\sqrt{x})^4}\,dx$

   Strategy: u-sub, $u = 2 + \sqrt{x}$ (rationalizes the radical).

   Answer: $-\dfrac{2}{3(2+\sqrt{x})^3} + C$.

7. $\displaystyle\int \frac{\sin^5 x}{\sec^2 x}\,dx$

Strategy: rewrite as $\sin^5 x \cos^2 x$; odd power of sine, $u = \cos x$.

Answer:
$$-\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C.$$

8. $\displaystyle\int \frac{3}{x^2+6x+9}\,dx$

Strategy: denominator is $(x+3)^2$; simple u-sub, $u = x+3$.

Answer: $-\dfrac{3}{x+3} + C$.

9. $\displaystyle\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$

Strategy: u-sub $u = \sqrt{x}$, then odd power of sine.

Answer:
$$-2\cos(\sqrt{x}) + \frac{2}{3}\cos^3(\sqrt{x}) + C.$$

10. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$

Strategy: u-sub $w = e^x$ turns it into a rational function; then partial fractions.

Answer: $\dfrac{1}{3}\ln\left|\dfrac{2e^x-1}{e^x+1}\right| + C$.

11. $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}\,dx$

Strategy: simplify first — this equals $\cos(2x)$.

Answer: $\dfrac{1}{2}\sin(2x) + C$.

12. $\displaystyle\int \frac{x+5}{x^2+4}\,dx$

Strategy: split into two pieces — a u-sub piece and an arctangent piece.

Answer:
$$\frac{1}{2}\ln(x^2+4) + \frac{5}{2}\arctan\left(\frac{x}{2}\right) + C.$$

2 Part 2: Evaluate

Evaluate the integrals below, and then pick 2 of the previous integrals (ones that you feel like you need more practice with) and evaluate them.

1. $\int \sqrt{7-6x-x^2}\,dx$

   Solution: Complete the square: $7-6x-x^2 = 16-(x+3)^2$. With $u = x+3$,

   $$\int \sqrt{16-u^2}\,du = \frac{u}{2}\sqrt{16-u^2} + 8\arcsin\left(\frac{u}{4}\right) + C.$$

   Substituting back,

   $$\int \sqrt{7-6x-x^2}\,dx = \frac{x+3}{2}\sqrt{7-6x-x^2} + 8\arcsin\left(\frac{x+3}{4}\right) + C.$$

2. $\int x^5 e^{x^2}\,dx$

   Solution: Let $t = x^2$, so $dt = 2x\,dx$ and $x^4 = t^2$. Then $x^5 e^{x^2} dx = t^2 e^t \cdot \frac{dt}{2}$, so

   $$\int x^5 e^{x^2}\,dx = \frac{1}{2}\int t^2 e^t\,dt = \frac{1}{2}e^t(t^2-2t+2) + C = \frac{1}{2}e^{x^2}(x^4-2x^2+2) + C.$$

3. $\int x^2\cos(5x)\,dx$

   Solution: Apply integration by parts twice:

   $$\int x^2\cos(5x)\,dx = \frac{x^2}{5}\sin(5x) + \frac{2x}{25}\cos(5x) - \frac{2}{125}\sin(5x) + C.$$

4. $\displaystyle\int \frac{\sec^2 x}{\cot^4 x}\,dx$

Solution: Since $\dfrac{1}{\cot^4 x} = \tan^4 x$, this is $\displaystyle\int \tan^4 x \sec^2 x\,dx$. Let $u = \tan x$, $du = \sec^2 x\,dx$:

$$\int \tan^4 x \sec^2 x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\tan^5 x}{5} + C.$$

5. Choice 1

This is a problem of your choosing from Part 1 — use the strategy identified there, and check your final answer against the Part 1 key.

6. Choice 2

This is a problem of your choosing from Part 1 — use the strategy identified there, and check your final answer against the Part 1 key.

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Given an integral, recognize which integration technique(s) — u-sub, integration by parts, trig integral identities, trig substitution, or partial fractions — will work, without necessarily evaluating it.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:

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2. **Confidently carry out the evaluation once a strategy has been chosen, including integrals that combine more than one technique.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:
