In this section, you will study three additional categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as $\sin(ku)$ and $\cos(ku)$. $\sin(2\theta), \cos(2\theta), \ldots$
2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$. $\sin^4 x, \cos^5 x, \ldots$ powers
3. The third category involves *functions of half-angles* such as $\sin\left(\frac{u}{2}\right)$.

1 Double-Angle Formulas

The double-angle formulas allow us to express trigonometric functions of twice an angle in terms of functions of the original angle.

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin(\underset{\theta}{u} + \underset{\theta}{v}) = \sin u \cos v + \cos u \sin v$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \boxed{2\cos^2\theta - 1} \\ &= \boxed{1 - 2\sin^2\theta}\end{aligned}$$

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}}$$

ex: $\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

$= 2\sin\theta\cos\theta$

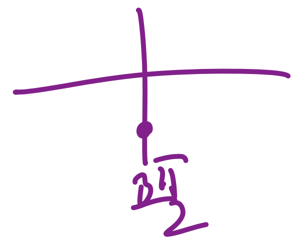
Example 1 Solving a Multiple-Angle Equation

$x = ?$

Solve $2\cos x + \sin(2x) = 0$.

Solution:

$\sin(2x) = 2\sin x\cos x$



$2\cos x + 2\sin x\cos x = 0$

$\underbrace{2\cos x}_{=0}\,(\underbrace{1+\sin x}_{=0}) = 0$

$\cos x = 0$

$x = \pi/2 + 2n\pi$

$x = \frac{3\pi}{2} + 2n\pi$

$1 + \sin x = 0$

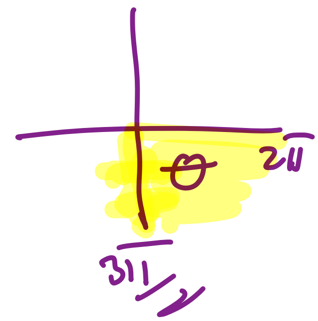
$\sin x = -1$

$x = \frac{3\pi}{2} + 2n\pi$

Example 2 Evaluating Functions Involving Double Angles

Use the following to find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

$$\cos\theta = \frac{5}{13}, \quad \frac{3\pi}{2} < \theta < 2\pi$$



Solution:

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\underbrace{\sin\theta}_{?}\left(\frac{5}{13}\right)$

$= 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{-120}{169}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \left(\frac{5}{13}\right)^2$

$\sin^2\theta = \frac{169}{169} - \frac{25}{169}$

$\sin^2\theta = \frac{144}{169}$

$\sin\theta = \pm\frac{12}{13}$

θ is in $\overline{IV}$

$\sin\theta = -\frac{12}{13}$

$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{5}{13}\right)^2 - 1$

$= -119/169$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(-12/5)}{1-(-12/5)^2} = \frac{120}{119}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-12}{13} \div \frac{5}{13} = \frac{-12}{\cancel{13}} \times \frac{\cancel{13}}{5} = \frac{-12}{5}$

2 Power-Reducing Formulas

The power-reducing formulas express powers of sine and cosine in terms of cosines of multiple angles.

$\cos(2\theta) = 1 - 2\sin^2\theta$
solve $\sin^2\theta$.

$$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

$$\boxed{\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

$2(2x)$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

Example 4 Reducing a Power

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution:

$$\sin^4 x = \sin^2 x \, \sin^2 x$$

$$= \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)$$

$$= \frac{1-\cos(2x)-\cos(2x)+\cos^2 2x}{4}$$

$$= \frac{1-2\cos(2x)+\cos^2(2x)}{4}$$

$$= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos(4x)}{2}\right)$$

$$= \frac{1}{4}\left(\frac{2-4\cos 2x+1+\cos 4x}{2}\right)$$

3 Half-Angle Formulas

The half-angle formulas express trigonometric functions of half an angle in terms of functions of the original angle.

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2}$

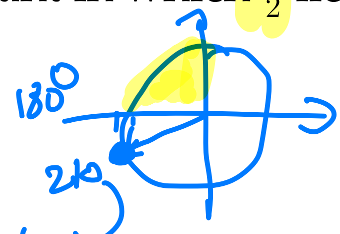
$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

The sign in the first two formulas depends on the quadrant in which $\frac{\theta}{2}$ lies.



Example 5 Using a Half-Angle Formula

Find the exact value of $\sin 105°$. $= \sin\left(\frac{210°}{2}\right)$

Solution:

$\theta = 210°$

$$\sin(105°) = \sin\left(\frac{210°}{2}\right) = +\sqrt{\frac{1-\cos(210°)}{2}}$$

$$= \sqrt{\frac{1-\left(-\sqrt{3}/2\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

Example 6 Solving a Trigonometric Equation

Find all solutions of $1 + \cos^2 x = 2\cos^2\left(\frac{x}{2}\right)$ in the interval $[0, 2\pi)$.

Solution:

$$1 + \cos^2 x = 2\left(\sqrt{\frac{1+\cos x}{2}}\right)^2$$

$$1 + \cos^2 x = \cancel{2}\left(\frac{1+\cos x}{\cancel{2}}\right)$$

$$\cancel{1} + \cos^2 x = \cancel{1} + \cos x$$

$$\cos^2 x - \cos x = 0$$

$$\cos x(\cos x - 1) = 0$$

$\cos x = 0$: $x = \pi/2$, $x = 3\pi/2$

$\cos x = 1$: $x = 0$