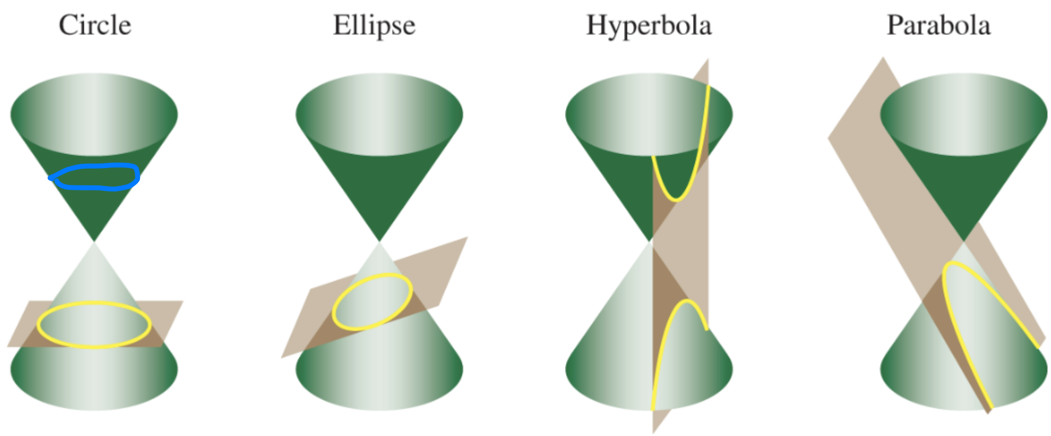
2. Ellipses

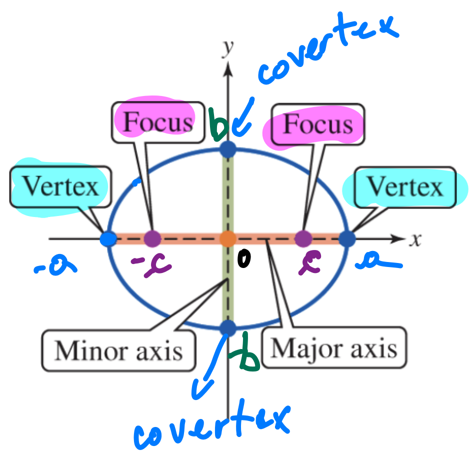
In this section we study ellipses from a geometric point of view.

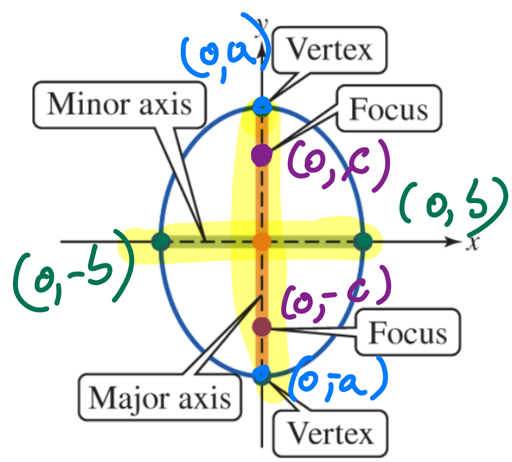


Definition: An ellipse is the set of all points in the plane such that the sum of the distance between a point (x, y) and two fixed points is a constant.

The fixed points are called foci (plural of focus) of the ellipse.

An ellipse may be elongated in any direction. We will study only those that are elongated horizontally and vertically.





orange : major axis
its length is 2a

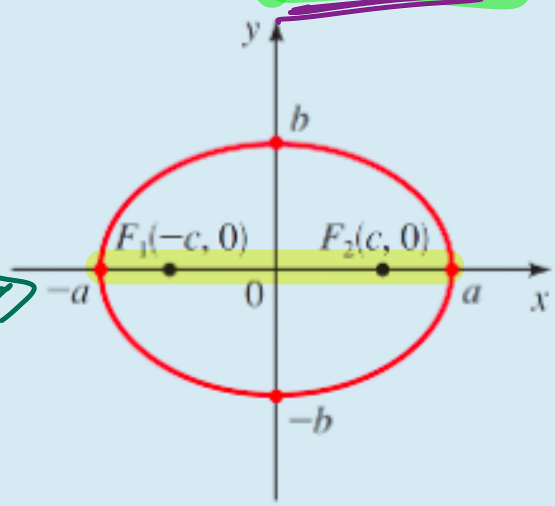
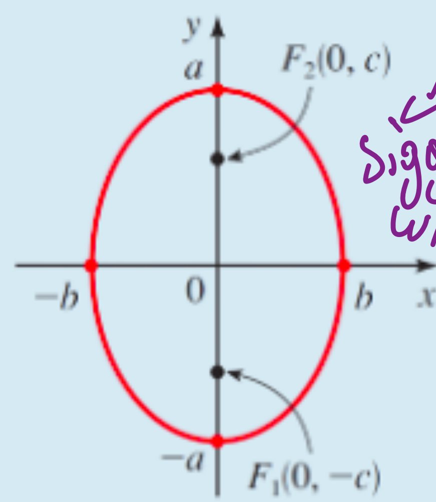
green : minor axis
its length is 2b

3. Equations and Graphs of ellipses

The following box summarizes the equation and features of an ellipse centered at the origin.

ELLIPSE WITH CENTER AT THE ORIGIN

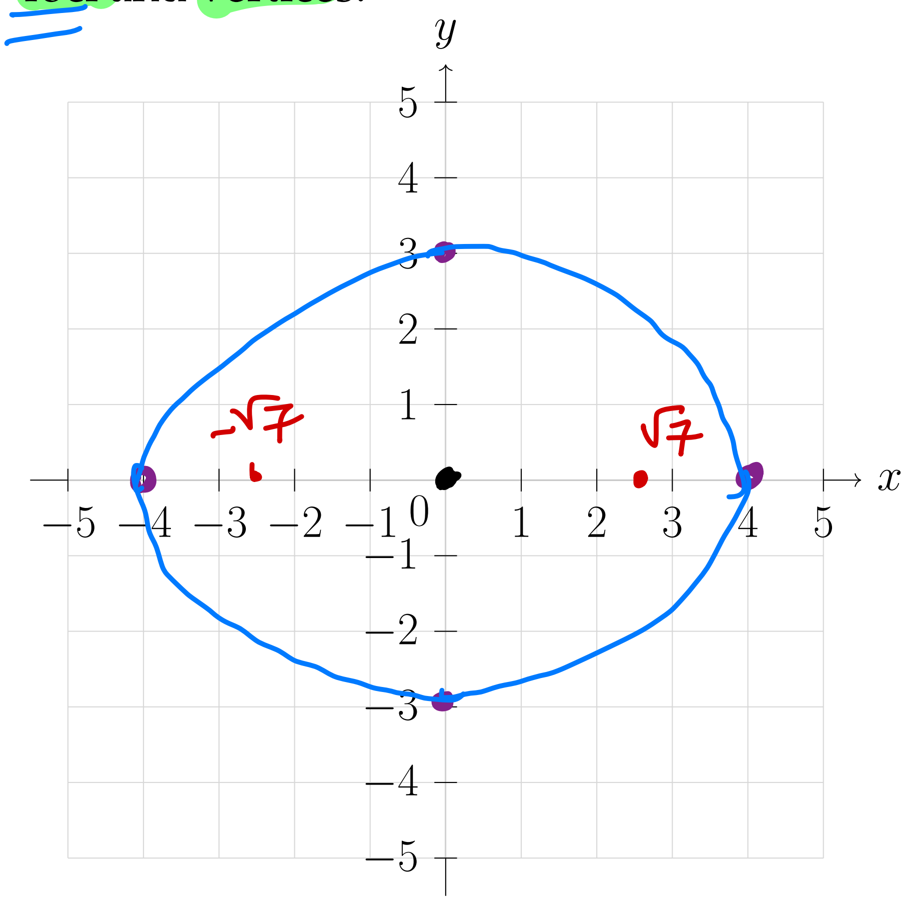
The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ $a > b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2 = a^2 - b^2$	$(0, \pm c)$, $c^2 = a^2 - b^2$
GRAPH		

bigger number with the x^2 term

bigger # w/ y^2

Example 5: Given $\boxed{\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1}$, graph the ellipse and identify the center, foci and vertices.



- elongated x axis

- foci :

- vertex : (4,0) 4
(-4,0)

- foci : $(\sqrt{7},0), (-\sqrt{7},0)$

$c^2 = a^2 - b^2$

$= 16 - 9 = 7$

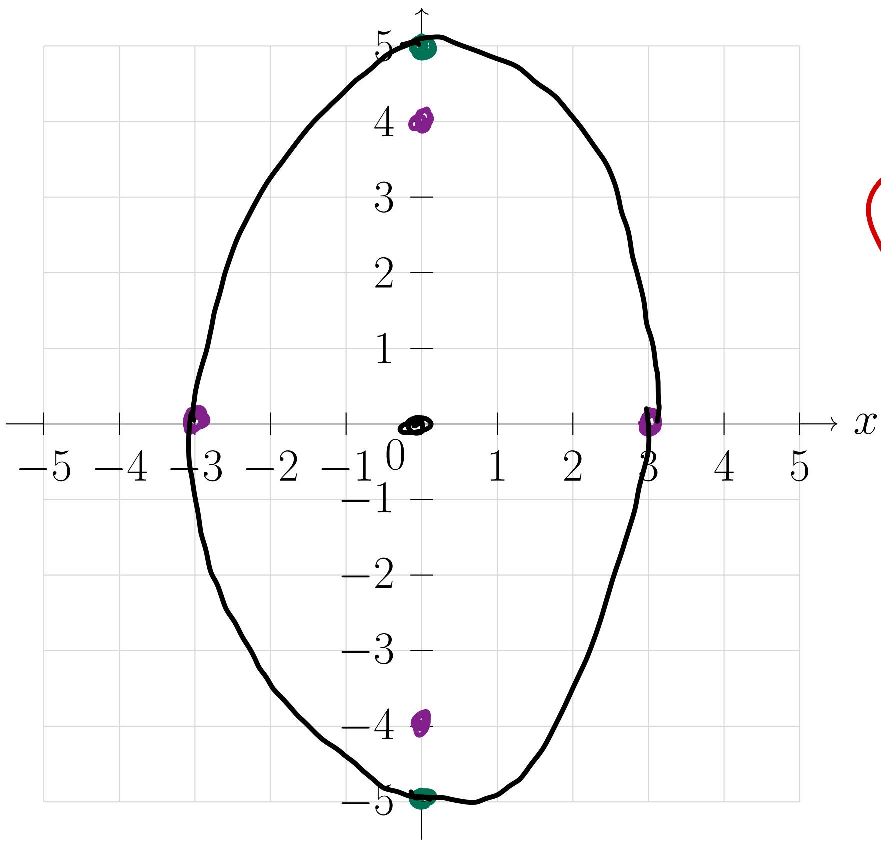
$c = \sqrt{7}$

$\dfrac{x^2}{4^2} + \dfrac{y^2}{3^2} = 1$ compare $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$a = 4 \rightarrow$ vertex

$b = 3 \rightarrow$ covertex

Example 6: Find the foci of the ellipse $25x^2 + 9y^2 = 225$, and sketch its graph.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

divide by 225

$$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

$$\boxed{\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1}$$

elongated along y axis

- vertices : $(0, a)$ & $(0,-5)$: $(0,5), (0,-5)$
- foci :

$$c^2 = a^2 - b^2$$

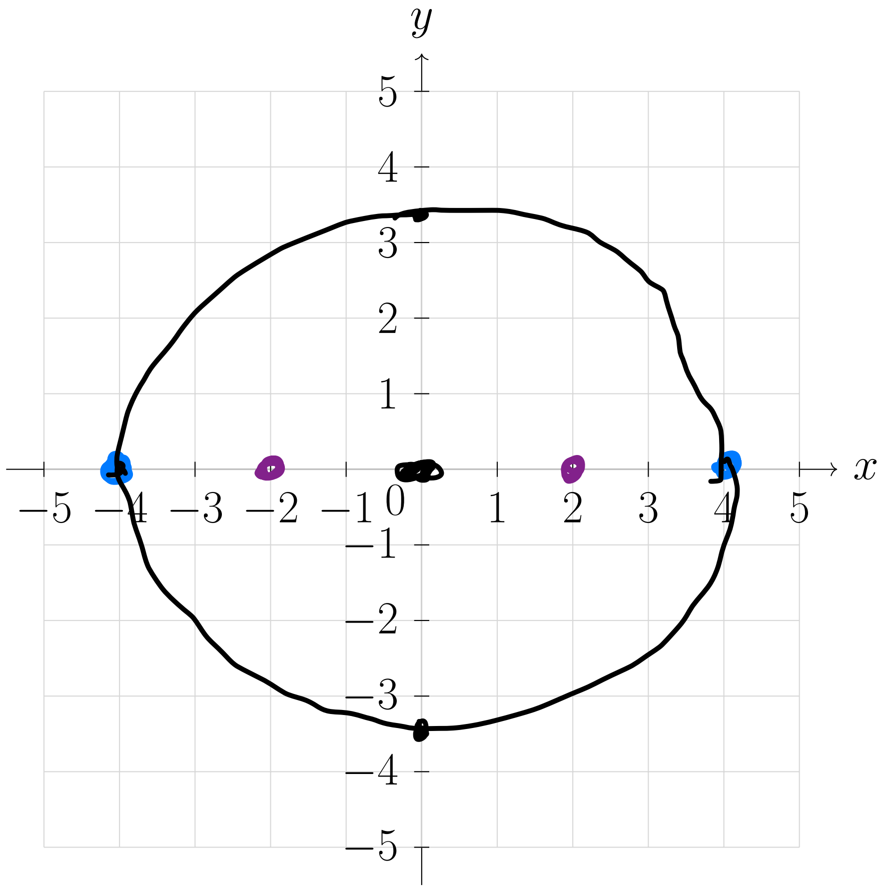
$$c^2 = 5^2 - 3^2 = 25 - 9 = 16$$

$$c = 4$$

$(5,0)$ & $(-5,0)$

- covertices : $(3,0)$ & $(-3,0)$

Example 7: The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find its equation, and sketch the graph.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a = 4$

$c = 2$

$c^2 = a^2 - b^2$

$4 = 16 - b^2$

$-12 = -b^2$

$b^2 = 12$

$b = \sqrt{12} \approx 3.46$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{12})^2} = 1$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{12} = 1} \rightarrow \frac{y^2}{12} = 1 - \frac{x^2}{16}$$

Calculator instructions

1) solve for y

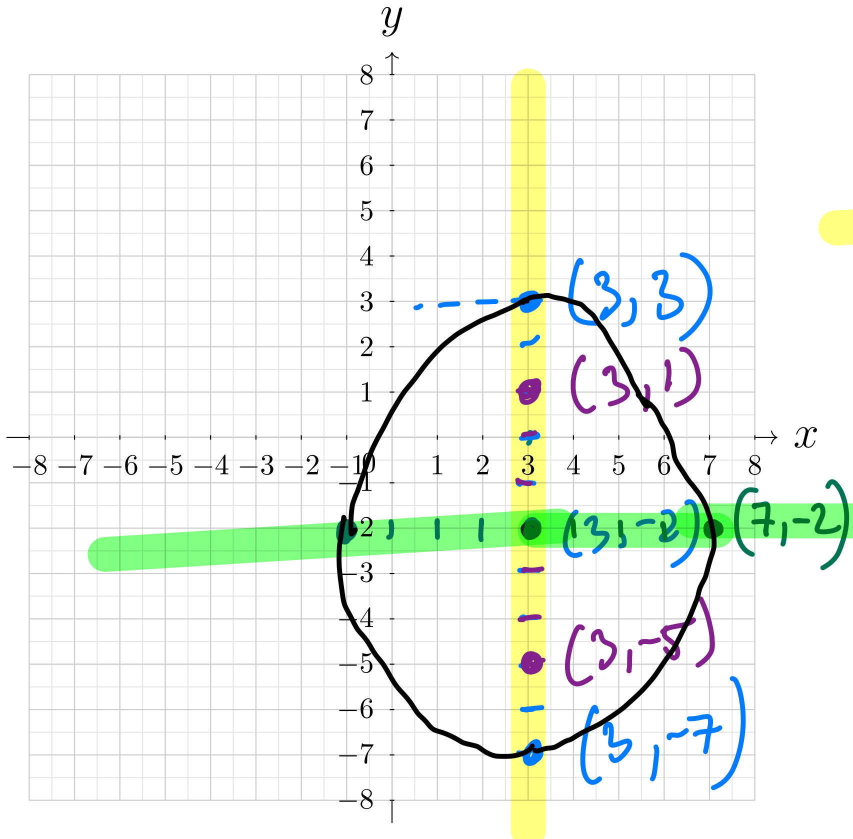
$$y^2 = 12\left(1 - \frac{x^2}{16}\right)$$

2) $y = \pm\sqrt{12\left(1 - \frac{x^2}{16}\right)}$

4. Graphing and Ellipse centered at (h, k)

Replacing x by $x - h$ and y by $y - k$ shifts the graph of an ellipse h units horizontally and k units vertically.

Example 8: Given $\dfrac{(x-3)^2}{16} + \dfrac{(y+2)^2}{25} = 1$, graph the ellipse and identify the center, foci and vertices.



center: $(3,-2)$

elongated vertically

$\boxed{a=5}$

$b=4$

vertex: $(3,3)$ & $(3,-7)$

foci: $(3,1)$ & $(3,-7)$

need c

$c^2=a^2-b^2$

$=25-16=9$

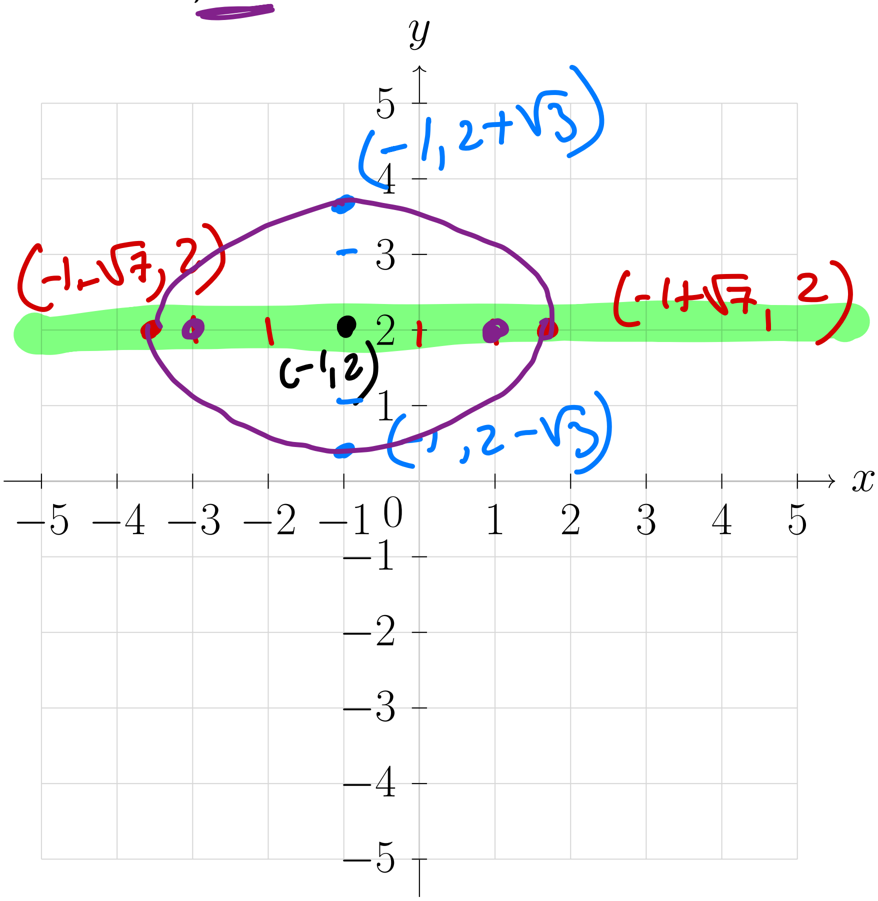
$c=3$

Example 9: Practice: Given $3x^2 + 7y^2 + 6x - 28y + 10 = 0$, write the equation of the ellipse in standard form, graph the ellipse and identify the center, foci and vertices.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



- elongated horizontally
- center $(-1, 2)$
- foci $c^2 = a^2 - b^2$
$= 7 - 3$
$= 4$

$$3x^2 + 7y^2 + 6x - 28y + 10 = 0$$

$$3x^2 + 6x + 7y^2 - 28y = -10$$

$$3(x^2 + 2x + 1) + 7(y^2 - 4y + 4) = -10 + 3 + 28$$

$$\frac{3(x+1)^2 + 7(y-2)^2}{21} = \frac{21}{21}$$

$$\boxed{\frac{(x+1)^2}{7} + \frac{(y-2)^2}{3} = 1}$$

$7 \to a = \sqrt{7}$

$3 \to b = \sqrt{3}$