

1 Compound Interest Models

quarterly $\Rightarrow n=4$
monthly $\Rightarrow n=12$

1.1 Compound Interest Formula with n Compounding Periods

Definition 1. For an initial principal P , annual interest rate r (expressed as a decimal), time t in years, and n compounding periods per year, the compound interest formula is:

$$A = P \left(1 + \frac{r}{n} \right)^{nt} \quad (1)$$

where A is the final amount.

Example 1. If you invest \$5,000 at an annual interest rate of 4% compounded quarterly for 10 years, find the final amount and how long it takes to reach \$9000.

$\Rightarrow P = 5000, \quad r = 4\% = 0.04, \quad n = 4$

a) Find A if $t = 10$ years

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$A = 5000 \left(1 + \frac{0.04}{4} \right)^{(4 \cdot 10)}$$

$$A = \$7,444.32$$

$$\ln\left(\frac{9}{5}\right) = \frac{nt \cdot \ln\left(1 + \frac{r}{n}\right)}{4 \ln\left(1 + \frac{0.04}{4}\right)}$$

$$\text{years } 14.77 = t$$

b) $t = ? \quad 9000 = 5000 \left(1 + \frac{0.04}{4} \right)^{4t}$

$$\ln \frac{9000}{5000} = \ln \left(1 + \frac{0.04}{4} \right)^{4t}$$

$$\ln \frac{9000}{5000} = 4t \ln \left(1 + \frac{0.04}{4} \right)$$

$$t = \frac{\ln \frac{9000}{5000}}{4 \ln \left(1 + \frac{0.04}{4} \right)} \quad \div 5000$$

1.2 Continuously Compounded Interest

Definition 2. As the number of compounding periods approaches infinity, we obtain the **continuously compounded interest** formula:

$$A = Pe^{rt}$$

(2)

where $e \approx 2.71828$ is the base of the natural logarithm.

Example 2. If you invest \$10,000 at an annual interest rate of 5% **compounded continuously** for 8 years, find the final amount and how long it takes for the amount to double. a

a) $P = 10,000$ $r = .05$, $t = 8 \text{ years}$

$$A = Pe^{rt}$$

$$A = 10,000 e^{(.05)(8)}$$

$$A = \$14,918$$

b)

$$A = Pe^{rt}$$

$$20,000 = 10,000 e^{.05t}$$

$$\ln 2 = \ln e^{.05t}$$

$$\ln 2 = .05t (\ln e) = 1$$

$$\ln 2 = .05t$$

$$\frac{\ln 2}{.05} = t$$

$$t = 13.86 \text{ years}$$

$$\log_e e = 1$$

2 Exponential Growth and Decay Models

2.1 Exponential Growth Model

Definition 3. The quantity Q at time t with initial value Q_0 and growth rate k is given by:

$$A = Pe^{rt}$$
$$Q = Q_0 e^{kt} \quad (\text{continuous}) \quad (3)$$

where $k > 0$ is the continuous growth rate. Q_0

Example 3. A bacterial colony starts with 500 bacteria and grows at a continuous rate of 15% per hour. a) How many bacteria will be present after 12 hours? b) How long it takes for the bacteria to reach 1500?

$$r = .15$$

a)

$$Q = Q_0 e^{kt}$$
$$Q = 500 e^{.15(12)}$$
$$Q = 3025 \text{ bacteria}$$

b)

$$Q = Q_0 e^{kt}$$
$$1500 = 500 e^{.15t}$$
$$\frac{1500}{500} = e^{.15t}$$
$$\ln 3 = \ln e^{.15t}$$
$$\ln 3 = .15t$$
$$\frac{\ln 3}{.15} = t$$
$$t = 7.32 \text{ years}$$

2.2 Exponential Decay Model

Definition 4. The quantity Q at time t with initial value Q_0 and decay rate k is given by:

$$Q = Q_0 e^{-kt} \quad (\text{continuous}) \quad (4)$$

where $k > 0$ is the continuous decay rate.

Example 4. A radioactive substance has an initial mass of 50 grams and decays at a continuous rate of 3% per day. a) What will be the mass after 30 days? b) How long until the mass is half of what it started?

$Q_0 = 50$

$k = .03$

a) $Q = Q_0 e^{-kt} \Rightarrow Q = 50 e^{-.03(30)} = 20.33 \text{ grams}$

b) $Q = Q_0 e^{-kt}$
 $25 = 50 e^{-.03t}$
 $\ln \frac{1}{2} = \ln e^{-.03t}$

$\ln(1/2) = -.03t$
 $\frac{\ln(1/2)}{-.03} = t$

$t = 23.1 \text{ days}$

Example 5. A population grows from 10,000 to 15,000 in 5 years. Find the continuous growth rate.

$k = ?$

$t_1: Q = Q_0 e^{kt}$
 $10000 = Q_0 e^{kt_1} \quad (2)$

$t_2: 15000 = Q_0 e^{kt_2} \quad (1)$

$t_2 - t_1 = 5$

$\frac{(1)}{(2)}: \frac{15000}{10000} = \frac{Q_0 e^{kt_2}}{Q_0 e^{kt_1}}$
 $\frac{3}{2} = e^{k(t_2 - t_1)}$
 $\frac{3}{2} = e^{k \cdot 5}$

$\ln \frac{3}{2} = \ln e^{5k}$
 $\ln(3/2) = 5k$
 $\frac{\ln(3/2)}{5} = k$
 $k = 7.1\%$