MTH 146 – Calculus II

Exam 2 Study Guide

Days 9–17: Sections 7.8, 11.1–11.9

What is on Exam 2

- Improper integrals of Type II (7.8)
- Sequences: limits, monotonic & bounded (11.1)
- Series: partial sums, geometric, telescoping, harmonic, Test for Divergence (11.2)
- Integral Test and p-series (11.3)
- Direct and Limit Comparison Tests (11.4)
- Alternating Series Test and error estimate (11.5)
- Absolute vs. conditional convergence (11.5)
- Ratio and Root Tests (11.6)
- Strategy for choosing a test (11.7)
- Power series: radius & interval of convergence (11.8)
- Functions as power series: substitute, multiply, differentiate, integrate (11.9)

How to use this guide: For each day, make sure you can (1) state every boxed definition or test *from memory*, (2) redo the worked example without looking, and (3) avoid the mistakes in the red boxes.

Day 9 — Improper Integrals: Type II (7.8)

Type II improper integral (the integrand blows up)

1. f continuous on $(a, b]$, discontinuous at a: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
2. f continuous on $[a, b)$, discontinuous at b: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$
3. Discontinuity at c *inside* (a, b): $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ — **both** must converge.

If the limit exists (finite), the integral **converges**; if it is infinite or does not exist, it **diverges**.

Example (endpoint discontinuity). $\displaystyle\int_0^1 \frac{dx}{x^{2/3}} = \lim_{t\to 0^+}\left[3x^{1/3}\right]_t^1 = \lim_{t\to 0^+}\left(3 - 3t^{1/3}\right) = 3$. Converges.

Example (discontinuity inside). $\displaystyle\int_0^3 \frac{dx}{x-1}$. The integrand blows up at $x = 1$, which is between 0 and 3, so split at 1: $\displaystyle\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-}\Big[\ln|x-1|\Big]_0^t = \lim_{t\to 1^-}\ln|t-1| = -\infty$. One piece diverges, so the integral **diverges**.

Example (IBP + l'Hôpital). $\displaystyle\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\Big[x\ln x - x\Big]_t^1 = -1 - \lim_{t\to 0^+}(t\ln t - t) = -1$, since $t\ln t = \dfrac{\ln t}{1/t} \overset{\text{H}}{\to} \lim \dfrac{1/t}{-1/t^2} = \lim(-t) = 0$.

Watch out: Always look for vertical asymptotes *inside* the interval (zeros of a denominator, $\ln x$ at 0, $\ln x$ in a denominator at $x = 1$, $\tan x$ at $\pi/2$). Plugging in the endpoints and ignoring the discontinuity gives a wrong, finite answer.

Day 10 — Sequences (11.1)

Sequences and their limits

A **sequence** $\{a_n\}$ is a function whose domain is the positive integers. $\lim_{n\to\infty} a_n = L$ means the terms get as close to L as we like for n large. If L is a finite number the sequence **converges**; otherwise it **diverges**.

- **From functions:** if $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $a_n \to L$. *So you can use l'Hôpital's Rule on $f(x)$.*
- **Squeeze Theorem:** $a_n \le b_n \le c_n$ and $a_n, c_n \to L \;\Rightarrow\; b_n \to L$.
- **Absolute value:** if $|a_n| \to 0$, then $a_n \to 0$.
- **Continuity:** if $a_n \to L$ and f is continuous at L, then $f(a_n) \to f(L)$.
- $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$, $= 1$ if $r = 1$, diverges if $r \le -1$ or $r > 1$.

Monotonic Sequence Theorem

Every **bounded**, **monotonic** sequence converges. Check monotonic by: the sign of $a_{n+1} - a_n$, comparing a_{n+1}/a_n to 1, or the sign of $f'(x)$.

Factorials: $n! = n(n-1)\cdots 2\cdot 1$, $0! = 1$, $(n+1)! = (n+1)\,n!$, $(2n+2)! = (2n+2)(2n+1)(2n)!$.

Example (l'Hôpital). $a_n = \dfrac{\ln n}{n}$: $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim \frac{1/x}{1} = 0$. Converges to 0.

Example (1^∞ form). $a_n = \left(1 + \frac{2}{n}\right)^n$: $\ln a_n = \dfrac{\ln(1+2/n)}{1/n} \overset{\text{H}}{\to} 2$, so $a_n \to e^2$.

Example (Squeeze). $a_n = \dfrac{n!}{n^n} = \dfrac{1}{n}\cdot\dfrac{2\cdot 3\cdots n}{n\cdot n\cdots n} \le \dfrac{1}{n}$, and $0 < a_n$, so $a_n \to 0$.

Day 10–11 — Series, Geometric, Telescoping, Test for Divergence (11.2)

Convergence of a series

$s_n = a_1 + a_2 + \cdots + a_n$ is the nth **partial sum**. $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. If this limit is a finite number s, the series **converges** to s; otherwise it **diverges**.

Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad \text{if } |r| < 1; \qquad \text{diverges if } |r| \ge 1.$$

Memory aid: $\text{sum} = \dfrac{\text{first term}}{1 - \text{ratio}}$.

Test for Divergence (nth-Term Test)

If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist), then $\sum a_n$ **diverges**.
If $\lim a_n = 0$, the test says **nothing**.

Harmonic series: $\displaystyle\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ **diverges**, even though $\frac{1}{n} \to 0$.

Telescoping series: (1) partial fractions if needed, (2) write out s_n and cancel, (3) find $\lim s_n$.

Operations: if $\sum a_n$ and $\sum b_n$ converge, then $\sum ca_n = c\sum a_n$ and $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$.

Example (geometric). $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n}$: first term $\frac{9}{4}$, $r = \frac{3}{4}$, sum $= \dfrac{9/4}{1 - 3/4} = 9$.

Example (telescoping). $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)$; $s_n=1-\dfrac{1}{n+1}\to 1$. Sum $=1$.

Example (telescoping, diverges). $\displaystyle\sum\ln\frac{n}{n+1}=\sum\big(\ln n-\ln(n+1)\big)$; $s_n=-\ln(n+1)\to-\infty$. Diverges (even though the terms $\to 0$).

Watch out:

- **Sequence vs. series:** $\{a_n\}$ converges means the *terms* approach a number; $\sum a_n$ converges means the *partial sums* approach a number. Example: $a_n=\frac{n}{n+1}\to 1$, but $\sum\frac{n}{n+1}$ diverges.
- **Check the starting index** of a geometric series: $\displaystyle\sum_{n=1}^{\infty}ar^n=\frac{ar}{1-r}$, not $\frac{a}{1-r}$.
- $\lim a_n=0$ does **not** mean the series converges.

Day 12 — The Integral Test and p-Series (11.3)

The Integral Test

If f is **continuous, positive, and (eventually) decreasing** on $[1,\infty)$ and $a_n=f(n)$, then

$$\sum_{n=1}^{\infty}a_n \quad\text{and}\quad \int_1^{\infty}f(x)\,dx$$

either **both converge** or **both diverge**. Show decreasing with $f'(x)<0$.

p-Series

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ **converges** if $p>1$ and **diverges** if $p\le 1$.

Example. $\displaystyle\sum\frac{1}{n^2+1}$: $f'(x)=\frac{-2x}{(x^2+1)^2}<0$; $\displaystyle\int_1^{\infty}\frac{dx}{x^2+1}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$, finite, so the series converges.

Example (eventually decreasing). $\displaystyle\sum\frac{\ln n}{n}$: $f'(x)=\frac{1-\ln x}{x^2}<0$ for $x>e$; $\displaystyle\int_1^{\infty}\frac{\ln x}{x}\,dx=\lim\frac{(\ln t)^2}{2}=\infty$. Diverges.

Example. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$: with $u=\ln x$, $\displaystyle\int_2^{\infty}\frac{dx}{x\ln x}=\lim\Big[\ln(\ln x)\Big]_2^t=\infty$. Diverges.

Watch out: The Integral Test tells you *whether* a series converges, **not its sum**: $\sum\frac{1}{n^2}=\frac{\pi^2}{6}$, but $\int_1^{\infty}\frac{dx}{x^2}=1$. Also, always verify the three conditions before using the test.

Day 13 — The Comparison Tests (11.4)

Direct Comparison Test ($\sum a_n$, $\sum b_n$ **with positive terms**)

- If $a_n\le b_n$ and $\sum b_n$ **converges**, then $\sum a_n$ converges. (*smaller than finite is finite*)
- If $a_n\ge b_n$ and $\sum b_n$ **diverges**, then $\sum a_n$ diverges. (*bigger than infinite is infinite*)

Limit Comparison Test (**positive terms**)

If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=c$ where c is **finite and** $c>0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Choosing b_n: keep only the **dominant terms** (highest powers of n on top and bottom) to compare with a p-series; if you see r^n, compare with a geometric series. Useful facts: $0 \le \sin^2 n, \cos^2 n \le 1$, $-1 \le \sin n \le 1$, $\ln n < n$, $e^{c/n} > 1$ for $c > 0$.

Example (direct). $\sum \dfrac{\cos^2 n}{n^2+9}$: $\dfrac{\cos^2 n}{n^2+9} \le \dfrac{1}{n^2}$, a convergent p-series ($p=2$). Converges.

Example (direct). $\sum \dfrac{\ln k}{k}$: for $k \ge 3$, $\ln k > 1$, so $\dfrac{\ln k}{k} > \dfrac{1}{k}$; harmonic diverges, so the series diverges.

Example (limit). $\sum \dfrac{2n^2+3n}{\sqrt{5+n^5}}$: $a_n \approx \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$; take $b_n = \frac{1}{\sqrt{n}}$: $\dfrac{a_n}{b_n} = \dfrac{2+3/n}{\sqrt{1+5/n^5}} \to 2 > 0$. $\sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$), so the series diverges.

Watch out: If your series is *smaller than a divergent* series or *larger than a convergent* series, Direct Comparison tells you **nothing**. Example: $\frac{1}{2^n-1} > \frac{1}{2^n}$ points the wrong way; use Limit Comparison with $b_n = \frac{1}{2^n}$ instead (limit $= 1$, converges).

Day 15 — Alternating Series, Absolute Convergence, Ratio & Root Tests (11.5–11.6)

Alternating Series Test (AST)

For $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$ with $b_n > 0$: if (i) $b_{n+1} \le b_n$ (eventually decreasing) and (ii) $\lim_{n\to\infty} b_n = 0$, then the series **converges**.

Alternating Series Estimation Theorem

If the series satisfies the AST, then $|s - s_n| \le b_{n+1}$: the error is at most the **first term you left out**. "Correct to k decimal places" means error $< 0.5 \times 10^{-k}$ (e.g. 3 places: error < 0.0005).

Absolute and conditional convergence

- $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
- $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **Theorem:** absolutely convergent $\Rightarrow$ convergent.

Ratio Test and Root Test

Let $L = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ (Ratio) or $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (Root).

- $L < 1$: **absolutely convergent**
- $L > 1$ or $L = \infty$: **divergent**
- $L = 1$: **inconclusive** — use a different test (the Root Test will also give 1).

Ratio: best for **factorials** and **exponentials** c^n. Root: best when the **whole term is to the nth power**.

Example (conditional). $\sum \dfrac{(-1)^{n-1}}{n}$ converges by the AST ($\frac{1}{n+1} < \frac{1}{n}$, $\frac{1}{n} \to 0$), but $\sum \frac{1}{n}$ diverges: **conditionally convergent**.

Example (absolute). $\sum \dfrac{(-1)^{n-1}}{n^2}$: $\sum \frac{1}{n^2}$ converges ($p=2$): **absolutely convergent**.

Example (estimation). $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to 3 decimals: $b_7 = \frac{1}{5040} < 0.0005$, so $s \approx s_6 \approx 0.368$.

Example (Ratio). $\sum (-1)^n \dfrac{n^3}{3^n}$: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{1}{3}\left(1+\dfrac{1}{n}\right)^3 \to \dfrac{1}{3} < 1$. Absolutely convergent.

Example (Ratio). $\sum \dfrac{n^n}{n!}$: $\dfrac{a_{n+1}}{a_n} = \left(1+\dfrac{1}{n}\right)^n \to e > 1$. Divergent.

Example (Root). $\sum \left(\dfrac{2n+3}{3n+2}\right)^n$: $\sqrt[n]{|a_n|} = \dfrac{2n+3}{3n+2} \to \dfrac{2}{3} < 1$. Absolutely convergent.

Example (Root fails, TFD works). $\sum \left(\dfrac{n}{n+1}\right)^n$: root $\to 1$ (inconclusive), but $a_n = \dfrac{1}{(1+1/n)^n} \to \dfrac{1}{e} \neq 0$, so it diverges.

Watch out:

- The AST can only show *convergence*. If $b_n \not\to 0$, use the Test for Divergence to conclude divergence.
- The Ratio Test always gives $L = 1$ for p-series and for rational or algebraic a_n, so do not use it on those.
- Simplify factorials carefully: $\dfrac{(2n+1)!}{(2n+3)!} = \dfrac{1}{(2n+3)(2n+2)}$.

Strategy for Testing Series (11.7)

Choosing a test

Refer to the **Strategy for Testing Series** handout given in class.

Day 16 — Power Series (11.8)

Power series and convergence

A **power series centered at** a is $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$. It always converges at $x = a$. Exactly one of these is true:

(i) it converges only at $x = a$: $R = 0$, interval $\{a\}$;

(ii) it converges for all x: $R = \infty$, interval $(-\infty, \infty)$;

(iii) it converges for $|x-a| < R$ and diverges for $|x-a| > R$; the endpoints $a \pm R$ must be checked separately.

R is the **radius of convergence**; the set of all x where the series converges is the **interval of convergence**.

Four steps for R and the interval

1. Ratio Test: compute $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ (it depends on x). **2.** Set the limit < 1 and solve to get $|x-a| < R$. **3.** Plug in **each endpoint** and test the resulting number series (Test for Divergence, p-series, comparison, AST, ...). **4.** Write the interval with [ ] or ( ) at each end.

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$: ratio $\to |x-3| < 1 \iff 2 < x < 4$, $R = 1$. $x = 4$: $\sum \frac{1}{n}$ diverges. $x = 2$: $\sum \frac{(-1)^n}{n}$ converges (AST). **Interval** $[2, 4)$**.**

Example. $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$: ratio $\to 3|x| < 1$, $R = \frac{1}{3}$. $x = -\frac{1}{3}$: $\sum \frac{1}{\sqrt{n+1}}$ diverges ($p = \frac{1}{2}$). $x = \frac{1}{3}$: $\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges (AST). **Interval** $\left(-\frac{1}{3}, \frac{1}{3}\right]$**.**

Example. $\sum n!\,x^n$: ratio $= (n+1)|x| \to \infty$ for $x \neq 0$, so $R = 0$. $\sum \frac{x^n}{(2n)!}$: ratio $= \frac{|x|}{(2n+1)(2n+2)} \to 0$, so $R = \infty$.

Watch out: The Ratio Test **always** gives $L = 1$ at the endpoints, so you must use a *different* test there. Rewrite $|2x-1| < 5$ as $\left|x - \frac{1}{2}\right| < \frac{5}{2}$ to read off the center and R.

Day 17 — Representations of Functions as Power Series (11.9)

The one series to build everything from

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \qquad |x| < 1.$$

Strategy: rewrite f as $\dfrac{a}{1-r}$ (factor out constants and powers of x), then $\dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} ar^n$, valid for $|r| < 1$. Solving $|r| < 1$ for x gives the interval; no Ratio Test needed.

Term-by-term differentiation and integration

If $f(x) = \sum c_n (x-a)^n$ has radius $R > 0$, then on $(a-R, a+R)$:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \qquad \int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

Both have the **same radius** R (the endpoints may change). After integrating, **find C by plugging in $x = 0$.**

Which tool?

- **Substitute** for x: $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \sum (-1)^n x^{2n}, \ |x| < 1.$
- **Factor out a constant**: $\dfrac{1}{2+x} = \dfrac{1}{2} \cdot \dfrac{1}{1-(-x/2)} = \sum \dfrac{(-1)^n x^n}{2^{n+1}}, \ |x| < 2.$
- **Multiply by a power of x**: $\dfrac{x^3}{x+2} = \sum \dfrac{(-1)^n x^{n+3}}{2^{n+1}}, \ |x| < 2.$
- **Differentiate** when you see a *squared* denominator:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n, \qquad |x| < 1.$$

- **Integrate** when you see ln or $\tan^{-1}$ (their derivatives are geometric-type):

$$\ln(1+x) = \int \frac{dx}{1+x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1.$$