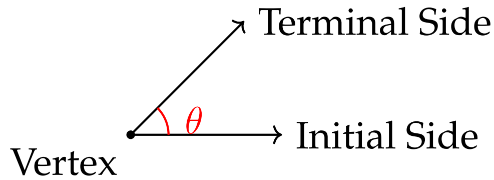
# 1 Angles

## 1.1 Basic Definitions

**Definition : Angle**

An **angle** is formed by two rays that share a common endpoint. The common endpoint is called the **vertex** of the angle, one ray is called the **initial side**, and the other ray is called the **terminal side**.




**Definition : Standard Position**

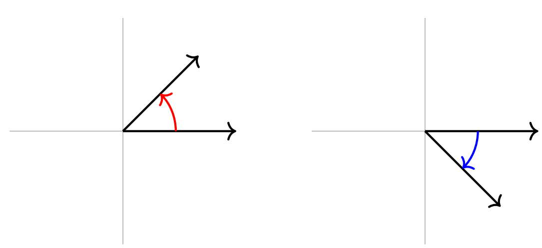
An angle is in **standard position** if:

- The vertex is at the origin of a rectangular coordinate system
- The initial side is along the positive $x$-axis

Notation: we use Greek letters to denote angles

**Definition : Positive and Negative Angles**

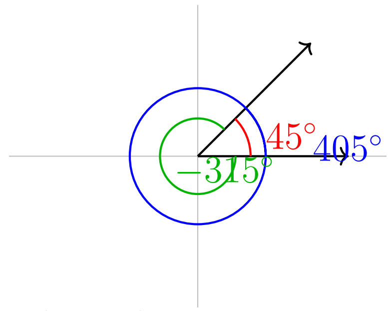
- A **positive angle** is generated by a ____________________ rotation from the initial side to the terminal side.
- A **negative angle** is generated by a ____________________ rotation from the initial side to the terminal side.



**Definition : Coterminal Angles**

**Coterminal angles** are angles in standard position that have ________________________________________.

If $\theta$ is an angle, then $\theta + 360°n$ where $n$ is an integer, is coterminal with $\theta$.




Coterminal Angles: $45°$, $405°$, and $-315°$

**Example 1: Finding Coterminal Angles**

Find two coterminal angles (one positive and one negative) for $\theta = 60°$.

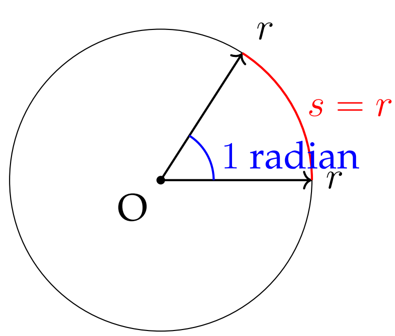
**Solution:**
For a positive coterminal angle, add $360°$:
For a negative coterminal angle, subtract $360°$:

# 2 Radian Measure

## 2.1 Definition of Radian

**Definition : Radian**

A **radian** is the measure of an angle that, when placed at the center of a circle, intercepts an arc equal in length to the radius of the circle.



**Definition : Central Angle**

A **central angle** is an angle whose vertex is at the center of a circle, with its sides being radii of the circle.

# 3 Degree and Radian Conversions

## 3.1 Converting Between Degrees and Radians

**Definition : Conversion Formulas**

To convert between degrees and radians, use the following relationships:

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$
$$1^\circ = \frac{\pi}{180} \text{ radians} \approx 0.01745 \text{ radians}$$

Therefore:

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$
$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

**Example 2: Converting from Degrees to Radians**

Convert the following angles from degrees to radians in terms of $\pi$: a) $45^\circ$ b) $120^\circ$ c) $-30^\circ$

**Solution:**

a) $45^\circ$ in radians:

b) $120^\circ$ in radians:

c) $-30^\circ$ in radians:

**Example 3: Converting from Radians to Degrees**

Convert the following angles from radians to degrees: a) $\frac{\pi}{3}$ b) $\frac{5\pi}{4}$ c) $-\frac{\pi}{2}$

**Solution:**

a) $\frac{\pi}{3}$ in degrees:

b) $\frac{5\pi}{4}$ in degrees:

c) $-\frac{\pi}{2}$ in degrees:

# 4 Complementary, Supplementary Angles, and Arc Length

## 4.1 Complementary and Supplementary Angles

**Definition : Complementary Angles**

Two angles are **complementary** if their sum equals $90^\circ$ or $\frac{\pi}{2}$ radians.
If angles $A$ and $B$ are complementary, then:

$$A + B = 90^\circ \text{or } A + B = \frac{\pi}{2} \text{ radians}$$

The **complement** of angle $A$ is $(90^\circ - A)$ or $(\frac{\pi}{2} - A)$ in radians.

**Definition : Supplementary Angles**

Two angles are **supplementary** if their sum equals $180^\circ$ or $\pi$ radians.
If angles $A$ and $B$ are supplementary, then:

$$A + B = 180^\circ \text{or } A + B = \pi \text{ radians}$$

The **supplement** of angle $A$ is $(180^\circ - A)$ or $(\pi - A)$ in radians.

**Example 4: Finding Complements and Supplements**

Find the complement and supplement of each angle: a) $25°$, b) $\frac{\pi}{4}$ radians, and c) $100°$

**Solution:**

a) For $\theta = 25°$:

$$\text{Complement} = $$
$$\text{Supplement} = $$

b) For $\theta = \frac{\pi}{4}$ radians:

$$\text{Complement} = $$
$$\text{Supplement} = $$

c) For $\theta = 100°$:

$$\text{Complement} = $$
$$\text{Supplement} = $$

### 4.2 Arc Length and Area of a Sector

**Definition : Arc Length and Area of a sector**

If a central angle $\theta$ in a circle of radius $r$ intercepts an arc, then the arc length $s$ is given by:

$$s = r\theta$$

where $\theta$ must be measured in radians. The area $A$ of a sector is given by:

$$A = \frac{1}{2}r^2\theta$$

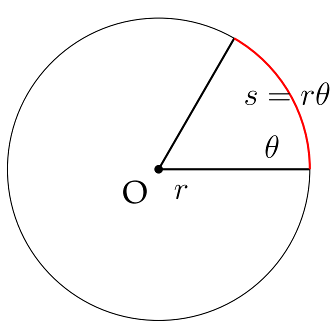


Figure 1: Arc length of a sector

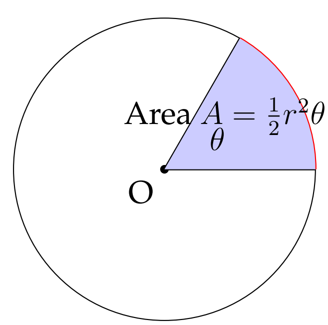


Figure 2: Area of a sector

**Example 5: Finding Arc Length**

Find the arc length intercepted by a central angle of $\frac{\pi}{3}$ radians in a circle with radius 12 cm.

**Solution:** Using the arc length formula with $r = 12$ cm and $\theta = \frac{\pi}{3}$ radians:

**Example 6: Finding Area of a Sector**

Find the area of a sector formed by a central angle of $72°$ in a circle with radius 5 m.

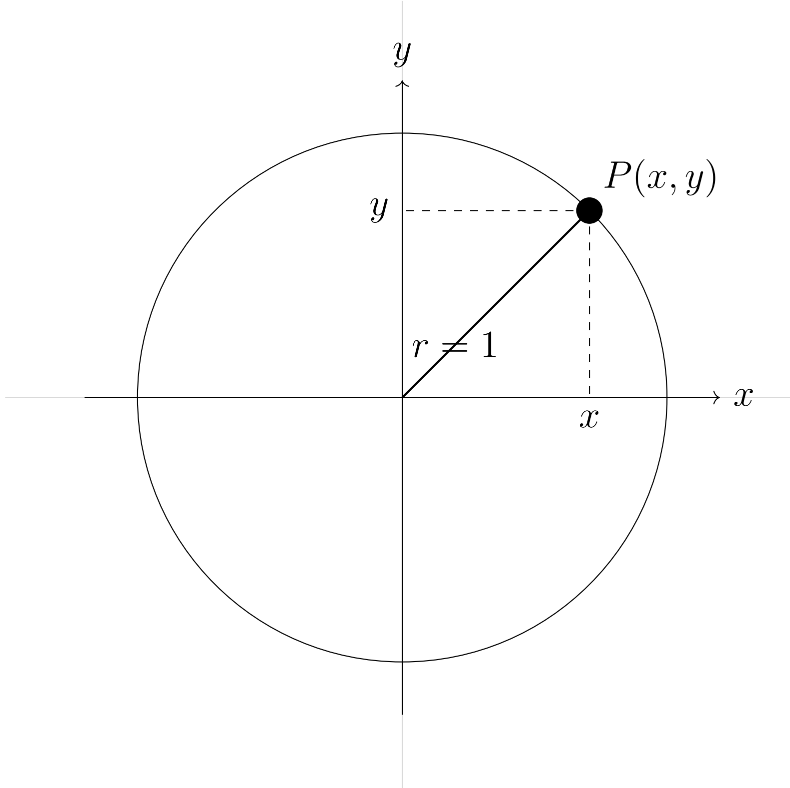
# 5 The Unit Circle and Trigonometric Functions

## 5.1 Definition of the Unit Circle

**Definition : Unit Circle**

The **unit circle** is a circle with radius 1 and center at the origin of the coordinate plane. Every point on the unit circle is at a distance of 1 unit from the origin.
The equation of the unit circle is:

$$x^2 + y^2 = 1$$




### 5.2 Six Trigonometric Functions: sine, cosine, tangent, cosecant, secant, cotangent

**Definition : Primary Trigonometric Functions**

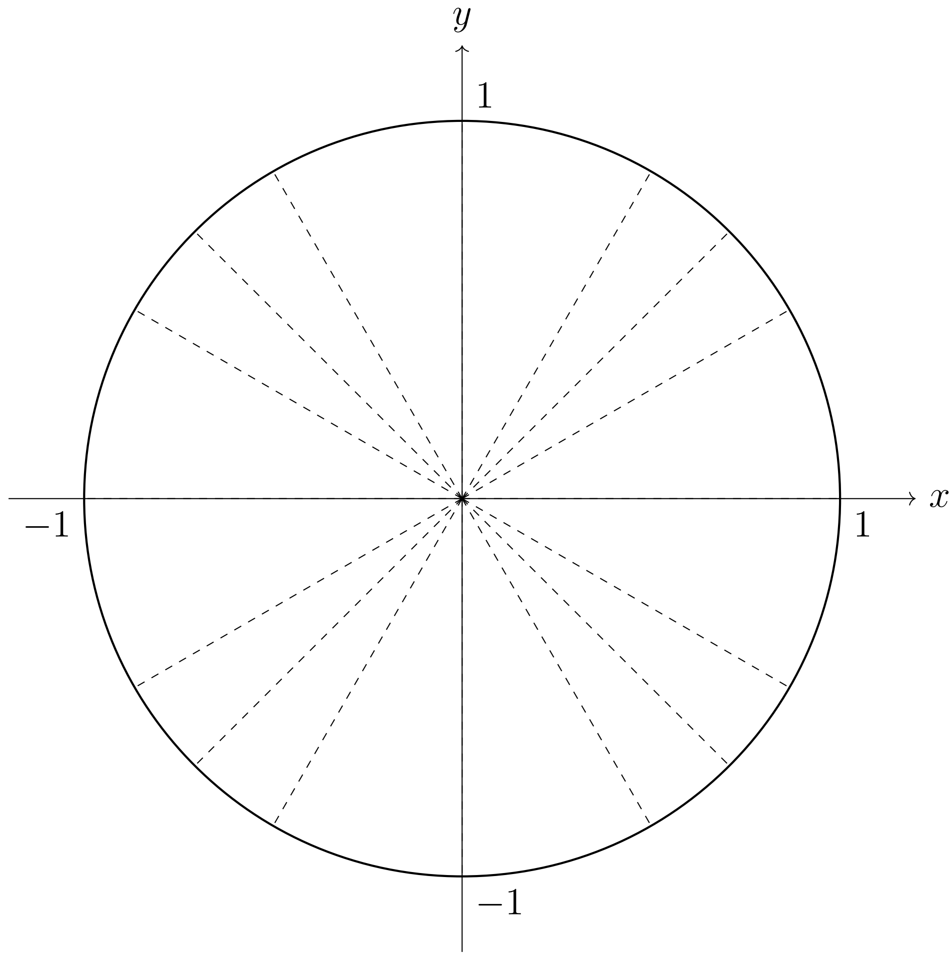
For a point $P(x, y)$ on the unit circle corresponding to an angle $\theta$ in standard position:

$$\begin{aligned} \sin\theta &= \\ \cos\theta &= \\ \tan\theta &= \qquad\qquad\quad , \quad \cos\theta \neq 0 \end{aligned}$$

**Definition : Reciprocal Trigonometric Functions**

The reciprocal trigonometric functions are defined as:

$$\begin{aligned} \csc\theta &= \qquad\qquad\quad , \quad \sin\theta \neq 0 \\ \sec\theta &= \qquad\qquad\quad , \quad \cos\theta \neq 0 \\ \cot\theta &= \qquad\qquad\qquad\qquad , \quad \sin\theta \neq 0 \end{aligned}$$



Unit Circle with Common Angles

### 5.3 Evaluating Trigonometric Functions

**Example 7: Evaluating Trig Functions for Special Angles**

Find the values of all six trigonometric functions for $\theta = 30^\circ$.

**Solution:**
For $\theta = 30^\circ = \frac{\pi}{6}$ radians, the point on the unit circle is:

$$P(\cos 30^\circ, \sin 30^\circ) = P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Therefore:
$\sin 30^\circ =$
$\cos 30^\circ =$
$\tan 30^\circ =$
$\csc 30^\circ =$
$\sec 30^\circ =$
$\cot 30^\circ =$

**Example 8: Finding Trig Functions from a Point**

If $P(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is a point on the unit circle corresponding to angle $\theta$, find all six trigonometric functions of $\theta$.

**Solution:**
Since the point is on the unit circle, we know that:

$$\cos\theta =$$
$$\sin\theta =$$

The other trigonometric functions can be found using these values:

$$\tan\theta =$$
$$\csc\theta =$$
$$\sec\theta =$$
$$\cot\theta =$$

### 5.4 Domains of Trigonometric Functions

**Definition : Domains of Trigonometric Functions**

The domains of the six trigonometric functions are:

$$\sin\theta : \text{All real numbers}$$
$$\cos\theta : \text{All real numbers}$$
$$\tan\theta : \text{All real numbers except } \theta = \frac{\pi}{2} + n\pi \text{ (where } n \text{ is an integer)}$$
$$\csc\theta : \text{All real numbers except } \theta = n\pi \text{ (where } n \text{ is an integer)}$$
$$\sec\theta : \text{All real numbers except } \theta = \frac{\pi}{2} + n\pi \text{ (where } n \text{ is an integer)}$$
$$\cot\theta : \text{All real numbers except } \theta = n\pi \text{ (where } n \text{ is an integer)}$$

### 5.5 Even and Odd Trigonometric Functions

**Definition : Even and Odd Functions**

A function $f$ is **even** if $f(-x) = f(x)$ for all $x$ in the domain. A function $f$ is **odd** if $f(-x) = -f(x)$ for all $x$ in the domain.

**Note : Even and Odd Trigonometric Functions**

## 5.6 Periodicity of Trigonometric Functions

**Definition : Periodic Functions**

A function $f$ is **periodic** if there exists a positive real number $p$ such that $f(x+p) = f(x)$ for all $x$ in the domain. The smallest such value of $p$ is called the **period** of the function.

**Note : Periods of Trigonometric Functions**

## 6 Summary of Key Concepts

1. **Angles** are formed by two rays with a common vertex; they have an initial side and a terminal side.
2. **Standard position** means the angle's vertex is at the origin and the initial side is along the positive x-axis.
3. **Positive angles** rotate counterclockwise; **negative angles** rotate clockwise.
4. **Coterminal angles** share the same terminal side and differ by multiples of $360°$ or $2\pi$ radians.
5. **Radians** and **degrees** are two common angle measures: $\pi$ radians $= 180°$.
6. **Complementary angles** sum to $90°$ or $\frac{\pi}{2}$ radians.
7. **Supplementary angles** sum to $180°$ or $\pi$ radians.
8. **Arc length** formula: $s = r\theta$ (where $\theta$ is in radians)
9. **Area of a sector** formula: $A = \frac{1}{2}r^2\theta$ (where $\theta$ is in radians)
10. **Unit circle** has radius 1 and is centered at the origin. Points on the unit circle are $(x, y) = (\cos\theta, \sin\theta)$.
11. **Six trigonometric functions**: sine, cosine, tangent, cosecant, secant, and cotangent.
12. **Even functions**: $\cos\theta$ and $\sec\theta$
13. **Odd functions**: $\sin\theta$, $\csc\theta$, $\tan\theta$, and $\cot\theta$
14. **Periodicity**: $\sin\theta$, $\cos\theta$, $\csc\theta$, and $\sec\theta$ have period $2\pi$; $\tan\theta$ and $\cot\theta$ have period $\pi$.