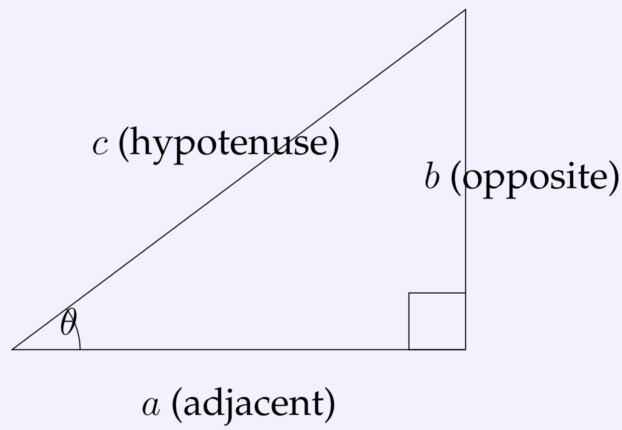
Right Triangle Trigonometry

Definition 4.3.1

In a right triangle with acute angle θ:



For right triangles, the Pythagorean Theorem applies: $c^2 = a^2 + b^2$, where c is the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$

Mnemonic: S=O/H C=A/H T=O/A

O: opposite A: adjacent

H: hyp.

Fundamental Identities

Reciprocal Identities

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sin(\theta) = \frac{1}{\csc(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cos(\theta) = \frac{1}{\sec(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$
$$\tan(\theta) = \frac{1}{\cot(\theta)}$$

Quotient Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$\theta = \frac{\pi}{6}$

$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6} = 1$

$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$

Pythagorean Identities

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

Even-Odd Properties

$$\left.\begin{aligned} \sec(-\theta) &= \sec(\theta) \\ \cos(-\theta) &= \cos(\theta) \end{aligned}\right\} \text{even}$$
$$\left.\begin{aligned} \sin(-\theta) &= -\sin(\theta) \\ \tan(-\theta) &= -\tan(\theta) \\ \csc(-\theta) &= -\csc(\theta) \\ \cot(-\theta) &= -\cot(\theta) \end{aligned}\right\} \text{odd}$$

Pythagorean identities Remark: $\cot^2\theta + 1 = \csc^2\theta$?

$\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$

$\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$

$1 + \cot^2\theta = \csc^2\theta$

Example 1. In the following examples find the exact values of the six trigonometric functions of the acute angle θ shown in the figure. Use the Pythagorean Theorem to determine the third side of the triangle.

$c^2 = a^2 + b^2$

Example Problem 1

Given: $c = 13$, $b = 5$.

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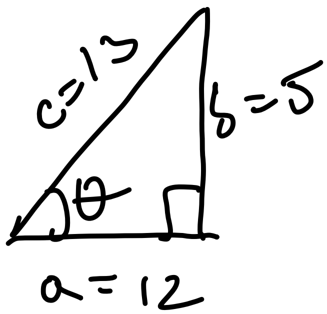


$\sin\theta = \frac{o}{h} = \frac{5}{13}$ $\qquad \csc\theta = \frac{13}{5}$

$\cos\theta = 12/13$ $\qquad \sec\theta = 13/12$

$\tan\theta = 5/12$ $\qquad \cot\theta = 12/5$

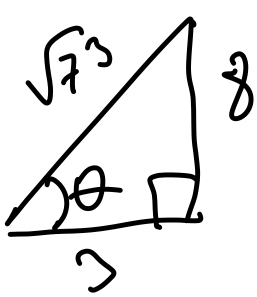
$c^2 = a^2 + b^2$

$13^2 = a^2 + 5^2$

$169 = a^2 + 25 \Rightarrow a^2 = 144$ $\quad a = 12$

Example Problem 2

Given: $a = 3$, $b = 8$.



$c = \sqrt{73}$

$\sin\theta = \frac{8}{\sqrt{73}} \cdot \frac{\sqrt{73}}{\sqrt{73}} = \frac{8\sqrt{73}}{73}$ $\qquad \csc\theta = \frac{\sqrt{73}}{8}$

$\cos\theta = \frac{3}{\sqrt{73}} \cdot \frac{\sqrt{73}}{\sqrt{73}} = \frac{3\sqrt{73}}{73}$ $\qquad \sec\theta = \frac{\sqrt{73}}{3}$

$\tan\theta = \frac{8}{3}$ $\qquad \cot\theta = \frac{3}{8}$

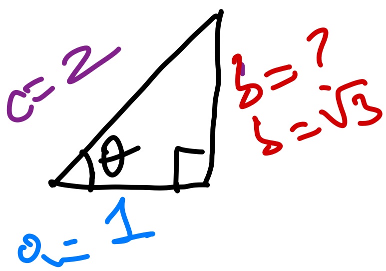
$\frac{4}{8} = \frac{1}{2}$

Example 2. Sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side of the triangle and then find the values of the other five trigonometric functions of θ.

Example Problem 1

Given: $\sec(\theta) = 2$

$c^2 = a^2 + b^2 \Rightarrow 4 = 1 + b^2 \Rightarrow b = \sqrt{3}$



$\sin\theta = \sqrt{3}/2$ $\qquad \csc\theta = 2/\sqrt{3} = \frac{2\sqrt{3}}{3}$

$\cos\theta = 1/2$ $\qquad \sec\theta = 2$

$\tan\theta = \sqrt{3}/1$ $\qquad \cot\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sec\theta = 2$

$\frac{1}{\cos\theta} = 2$

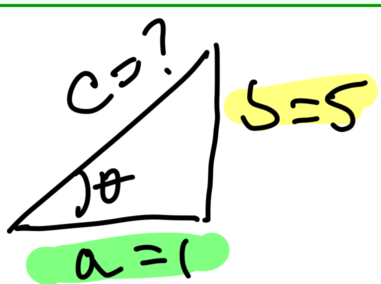
$\boxed{\cos\theta = \frac{1}{2}}$

$\frac{adj}{hyp} = \frac{1}{2} \Rightarrow$ adj $= 1$, hyp $= 2$

Example Problem 2

Given: $\tan(\theta) = 5$

$c^2 = a^2 + b^2 = 1 + 25 = 26$



$\tan\theta = 5$

$\frac{op}{adj} = \frac{5}{1}$

$c = \sqrt{26}$

$\sin\theta = \frac{5}{\sqrt{26}}$ $\qquad \csc\theta = \frac{\sqrt{26}}{5}$

$\cos\theta = \frac{1}{\sqrt{26}}$ $\qquad \sec\theta = \sqrt{26}$

$\cot\theta = \frac{1}{5}$

One more example

$\boxed{\theta = 2\pi/3}$

$\sin\theta = \sqrt{3}/2$

$\cos\theta = -1/2$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}/2}{-1/2} = \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{-\cancel{2}}{1} = -\sqrt{3}$

$\csc\theta = 1/\sin\theta = 2/\sqrt{3} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\sec\theta = \frac{1}{\cos\theta} = -2$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{-3} = \frac{-\sqrt{3}}{3}$

$= -\frac{\sqrt{3}}{3}$

Cofunction Identities

Cofunctions of complementary angles (sum to $90^\circ or \pi_2$ radians) are equal.

co function

$$\sin(\theta) = \cos(90^\circ - \theta) \quad \text{complementary} \quad \cos(\theta) = \sin(90^\circ - \theta)$$
$$\tan(\theta) = \cot(90^\circ - \theta) \qquad \cot(\theta) = \tan(90^\circ - \theta)$$
$$\sec(\theta) = \csc(90^\circ - \theta) \qquad \csc(\theta) = \sec(90^\circ - \theta)$$

$\sin(30^\circ) = \cos(90 - 30^\circ)$

Example 3. Use the function value(s) and the trigonometric identities to evaluate each trigonometric function.

Example Problem 1

Given: $\sin(30^\circ) = \frac{1}{2}$, $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.

(a) $\tan(30^\circ) = 1/2 \div \sqrt{3}/2 = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

(b) $\sin(60^\circ) = \cos(30^\circ) = \sqrt{3}/2$

(c) $\cos(60^\circ) = \sin(30^\circ) = 1/2$

(d) $\cot(30^\circ) = \frac{1}{\tan 30^\circ} = \sqrt{3}$

$\csc\theta = \frac{1}{\sin\theta}$

Example Problem 2

Given: $\csc(\theta) = 5$, $\sec(\theta) = \frac{5\sqrt{6}}{12}$

$\sec\theta = \frac{1}{\cos\theta}$

(a) $\sin(\theta) = 1/5 = \frac{1}{\csc\theta}$ $\qquad \csc\theta = \frac{1}{\sin\theta}$

(b) $\cos(\theta) = \frac{12}{5\sqrt{6}}$ $\qquad \cos\theta = \frac{1}{\sec\theta} = \frac{12}{5\sqrt{6}}$

(c) $\tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{1}{5} \div \frac{12}{5\sqrt{6}} = \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}\sqrt{6}}{12} = \frac{\sqrt{6}}{12}$

(d) $\sec(90^\circ - \theta) = \csc\theta = 5$

Example Problem 3

Given: $\cot(\theta) = 3$

Check your answer:

$\cot\theta = \frac{adj}{opp} = \frac{3}{1} \Rightarrow$ adj $= 3 = a$, opp $= 1 = b$

so $c^2 = 3^2 + 1^2$

$\boxed{c = \sqrt{10}}$

$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{3}$

$\sin\theta = \frac{opp}{hyp} = \frac{1}{\sqrt{10}}$

$\cos\theta = adj/hyp = 3/\sqrt{10}$

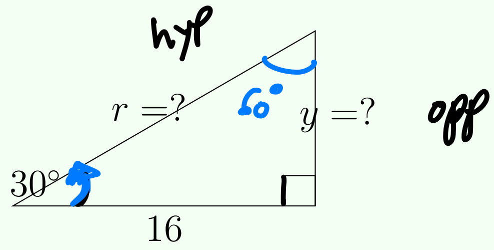
$\csc\theta = \sqrt{10}$

$\sec\theta = \sqrt{10}/3$

Example 4. Find the exact values of the indicated variables.

Example Problem 1

Find y and r in a right triangle with angle 30°, adjacent side $= 16$



$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\frac{16}{r} = \frac{\sqrt{3}}{2} \qquad 32 = r\sqrt{3} \implies \boxed{r = \frac{32}{\sqrt{3}}}$$

Find y:

Method 1 Pyt. Theorem

$$16^2 + y^2 = \left(\frac{32}{\sqrt{3}}\right)^2 \implies y = ?$$

Method 2

$$\tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$\frac{\text{opp}}{\text{adj}} =$$

$$\frac{y}{16} = \frac{\sqrt{3}}{3}$$

$$y = ?$$

Method 3:

$$\cos 60^\circ = \frac{y}{32/\sqrt{3}} \implies y = ?$$