Identity vs. Conditional Equation

A trigonometric identity is an equation involving trigonometric functions that is **true for all values in the domain of the variable**. Unlike a conditional equation, which is only true for specific values of the variable, an identity is true for all possible inputs where the expressions are defined.
For example, $\sin^2\theta + \cos^2\theta = 1$ **is an identity** because it is true for all values of θ. In contrast, $\sin\theta = \frac{1}{2}$ is a **conditional equation** because it is only true for specific values of θ, such as $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ (to name a couple).

Purpose of Knowing Identities

- They help solve trigonometric equations by converting them into more manageable forms.
- They enable us to integrate or differentiate complicated functions involving trigonometric terms.

Guidelines for Verifying Identities

1. Work with one side of the equation at a time, typically the more complex side.
2. Never perform operations that assume the identity is true (like cross-multiplication).
3. Look for opportunities to use fundamental identities to simplify expressions.
4. Convert all functions to sines and cosines when dealing with multiple different functions.
5. Factor expressions when possible.

6. Look for common denominators when dealing with fractions.
7. Remember that verification is complete when one side is transformed to match the other.

Examples of Verifying Trigonometric Identities

Used $\sin^2\theta+\cos^2\theta=1$
$\sec\theta=\frac{1}{\cos\theta}$

Example 1: $\dfrac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\cdot\sec^2\theta}=\boxed{1}$

Solution:

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\cdot\sec^2\theta}=\frac{1}{\cos^2\theta\ \sec^2\theta}=\frac{1}{\cancel{\cos^2\theta}\,\frac{1}{\cancel{\cos^2\theta}}}=$$

$$=\frac{1}{1}=\boxed{1}$$

Example 2: $2\csc^2\beta=\underbrace{\dfrac{1}{1-\cos\beta}+\dfrac{1}{1+\cos\beta}}_{\text{more complicated}}$

Solution:

$$\boxed{\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}}\ \underset{\substack{\text{bring}\\ \text{to}\\ \text{common}\\ \text{denom}}}{=}\ \frac{1}{1-\cos\beta}\,\frac{1+\cos\beta}{1+\cos\beta}+\frac{1}{1+\cos\beta}\,\frac{1-\cos\beta}{1-\cos\beta}$$

$$=\frac{1+\cancel{\cos\beta}+1-\cancel{\cos\beta}}{(1-\cos\beta)(1+\cos\beta)}=\frac{2}{1+\cancel{\cos\beta}-\cancel{\cos\beta}-\cos^2\beta}$$

$$=\frac{2}{1-\cos^2\beta}=\frac{2}{\sin^2\beta}=2\cdot\frac{1}{\sin^2\beta}=\boxed{2\csc^2\beta}$$

used $\sin^2\beta+\cos^2\beta=1\Rightarrow\sin^2\beta=1-\cos^2\beta$
$\frac{1}{\sin\beta}=\csc\beta$.

Example 3: $(sec^2x - 1)(sin^2x - 1) = -\sin^2 x$

Solution:

$$(\sec^2 x - 1)(\sin^2 x - 1) = \tan^2 x \, (-\cos^2 x) = -\tan^2 x \cdot \cos^2 x$$

$$= -\frac{\sin^2 x}{\cancel{\cos^2 x}} \cdot \cancel{\cos^2 x}$$

$$= \boxed{-\sin^2 x}$$

Example 4: $\tan x + \cot x = \sec x \csc x$

Solution:

I: $$\tan x + \cot x = \frac{\tan x}{1} + \frac{1}{\tan x} = \frac{\tan x}{1}\frac{\tan x}{\tan x} + \frac{1}{\tan x}$$

$$= \frac{\tan^2 x + 1}{\tan x} = \frac{\sec^2 x}{\tan x} = \frac{\sec^2 x}{\frac{\sin x}{\cos x}} =$$

$$= \frac{\sec^2 x \cdot \cos x}{\sin x} = \frac{1}{\cos^{\cancel{2}} x} \cdot \frac{\cancel{\cos x}}{\sin x}$$

$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \ \csc x$$

II $$\tan x + \cot x = \frac{\sin x}{\cos x}\frac{\sin x}{\sin x} + \frac{\cos x}{\sin x}\frac{\cos x}{\cos x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} =$$

$$= \sec x \ \csc x$$

Example 5: $\sec x + \tan x = \frac{\cos x}{1-\sin x}$

Solution:

$$\frac{\cos x}{(1-\sin x)}\frac{(1+\sin x)}{(1+\sin x)} = \frac{\cos x + \cos x \sin x}{1+\cancel{\sin x}-\cancel{\sin x}-\sin^2 x} = \frac{\cos x + \cos x \sin x}{\cos^2 x}$$

$$= \frac{\cancel{\cos x}}{\cos^{\cancel{2}} x} + \frac{\cancel{\cos x}\sin x}{\cos^{\cancel{2}} x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \boxed{\sec x + \tan x}$$

Example 6: $\frac{\cot^2\theta}{1+\csc\theta} = \frac{1-\sin\theta}{\sin\theta}$

Strategy: work with each side separatly

Solution:

$= ☆ \qquad = ☆$

LHS

$$\frac{\cot^2\theta}{1+\csc\theta} = \frac{\csc^2\theta - 1^2}{1+\csc\theta} = \frac{(\csc\theta-1)\cancel{(\csc\theta+1)}}{\cancel{1+\csc\theta}}$$

$$= \boxed{\csc\theta - 1} ☆$$

RHS

$$\frac{1-\sin\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta}$$

$$= \boxed{\csc\theta - 1}$$

Check out other completed notes for a different method

$$\frac{\cot^2\theta}{1+\csc\theta}\frac{(1-\csc\theta)}{1-\csc\theta} = \frac{\cot^2\theta(1-\csc\theta)}{1-\csc^2\theta} = \frac{\cancel{\cot^2\theta}(1-\csc\theta)}{-\cancel{\cot^2\theta}}$$

$$= \csc\theta - 1 ☆$$