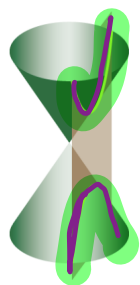


1. Hyperbolas

In this section we study hyperbolas from a geometric point of view. We begin with the geometric definition of a hyperbola. Although ellipses and hyperbolas have completely different shapes, their definitions and equations are similar.

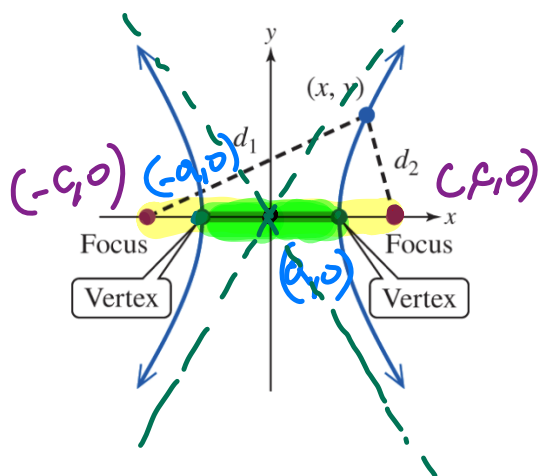
Instead of using the sum of distances from two fixed foci, as in the case of an ellipse, we use the **difference to define a hyperbola**.



Hyperbolas have 2 branches

GEOMETRIC DEFINITION OF A HYPERBOLA

A **hyperbola** is the set of all points in the plane, the **difference** of whose distances from **two fixed points F_1 and F_2** is a constant. (See Figure 1.) These two fixed points are the **foci** of the hyperbola.



orange: transverse axis
with length $2a$

2. Equations and Graphs of hyperbolas

The following box summarizes the equation and features of hyperbolas centered at the origin.

HYPERBOLA WITH CENTER AT THE ORIGIN

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a > 0, b > 0$$

VERTICES

$$(\pm a, 0)$$

TRANSVERSE AXIS

Horizontal, length $2a$

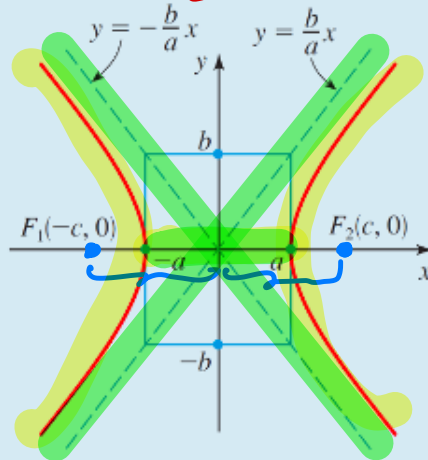
ASYMPTOTES

$$y = \pm \frac{b}{a}x$$

FOCI

$$(\pm c, 0), \quad c^2 = a^2 + b^2$$

GRAPH



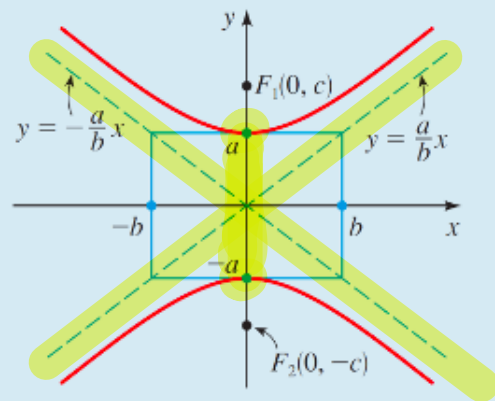
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad a > 0, b > 0$$

$$(0, \pm a)$$

Vertical, length $2a$

$$y = \pm \frac{a}{b}x$$

$$(0, \pm c), \quad c^2 = a^2 + b^2$$



3. Graphing hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

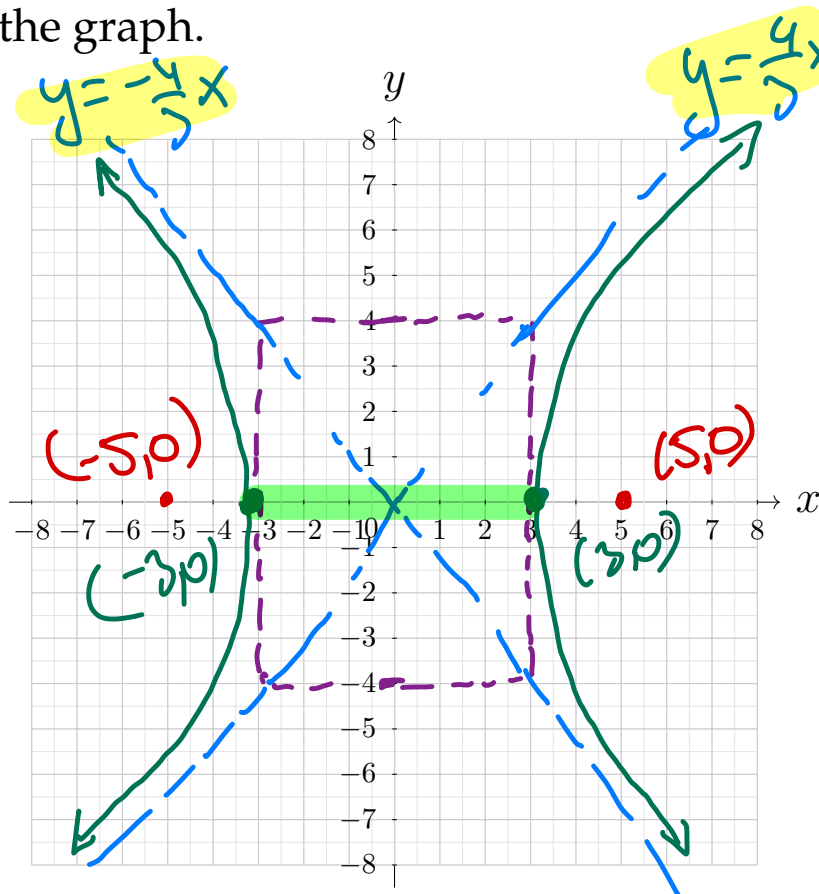
The following box summarizes the equation and features of hyperbolas centered at the origin.

HOW TO SKETCH A HYPERBOLA

→ reference rectangle

1. **Sketch the Central Box.** This is the rectangle centered at the origin, with sides parallel to the axes, that crosses one axis at $\pm a$ and the other at $\pm b$.
2. **Sketch the Asymptotes.** These are the lines obtained by extending the diagonals of the central box.
3. **Plot the Vertices.** These are the two x -intercepts or the two y -intercepts.
4. **Sketch the Hyperbola.** Start at a vertex, and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.

Example 1: A hyperbola has the equation $16x^2 - 9y^2 = 144$. Find the vertices, foci, length of the transverse axis, and asymptotes, and sketch the graph.



Divide by 144

$$\frac{16x^2}{144} - \frac{9y^2}{144} = \frac{144}{144}$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$$

$$a=3, b=4$$

Transverse axis: $2a = 2 \cdot 3 = 6$

foci: $(c, 0)$ & $(-c, 0)$
 $c^2 = a^2 + b^2 = 9 + 16 = 25$

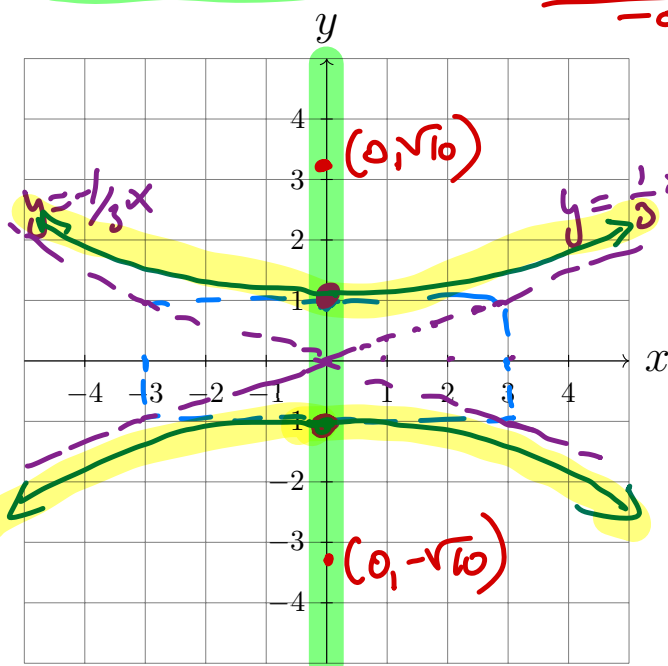
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Example 2: Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola, and sketch its graph.

$$x^2 - 9y^2 + 9 = 0$$

$$\Rightarrow \frac{x^2}{-9} - \frac{y^2}{-9} = -1 \Rightarrow \frac{x^2}{-9} + y^2 = 1$$

$$\Rightarrow y^2 - \frac{x^2}{9} = 1 \Rightarrow \frac{y^2}{1^2} - \frac{x^2}{3^2} = 1$$



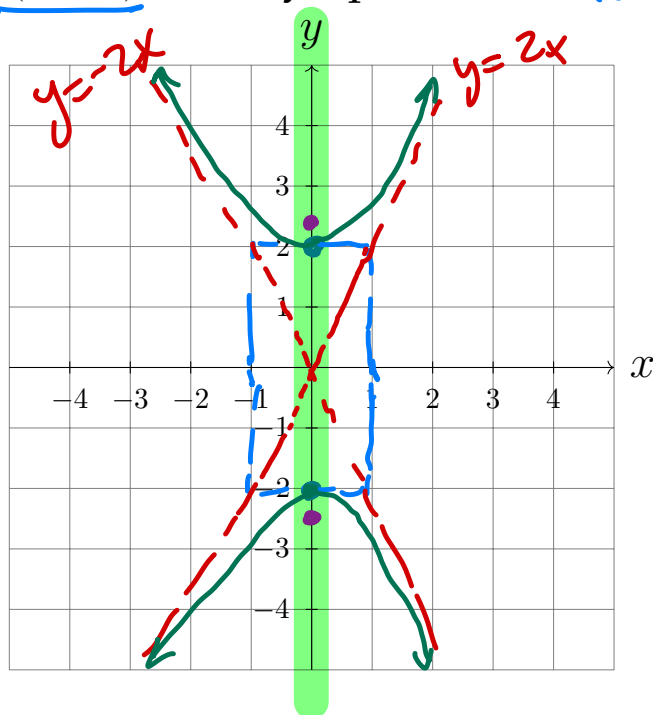
$$a=1, b=3$$

$$\text{transverse axis: } 2a = 2 \cdot 1 = 2$$

$$\text{foci: } c^2 = a^2 + b^2 = 1 + 9 = 10$$

$$(0, \sqrt{10}), (0, -\sqrt{10})$$

Example 3: Find the equation and the foci of the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm 2x$. Sketch the graph.



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$y = \pm \frac{a}{b} x \quad \left\{ \begin{array}{l} a=2 \\ b=1 \end{array} \right.$$

$$\text{foci: } c^2 = a^2 + b^2 = 4 + 1 = 5$$

$$\text{equation: } \frac{y^2}{2^2} - \frac{x^2}{1} = 1$$

$$\frac{y^2}{4} - x^2 = 1$$

4. Hyperbolas with center at (h, k)

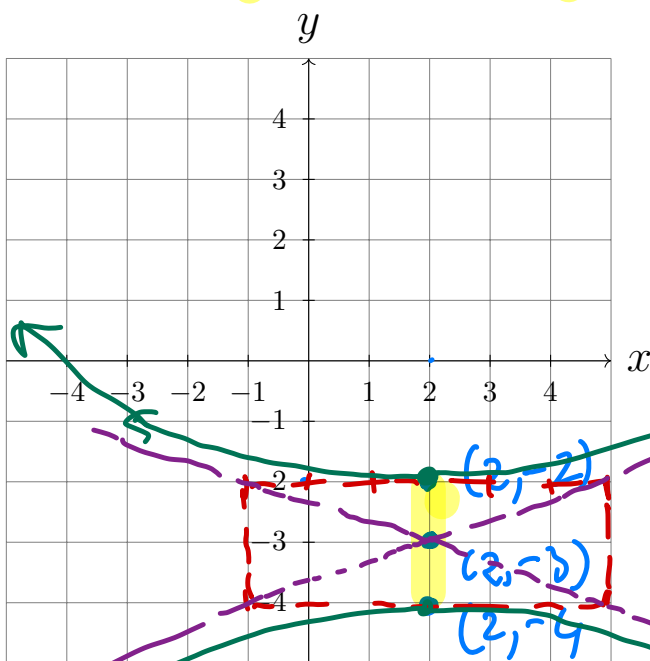
The following box summarizes the equation and features of hyperbolas centered at (h, k) .

Standard Forms of an Equation of a Hyperbola Centered at (h, k)

The standard forms of an equation of a hyperbola centered at (h, k) are as follows. Assume that $a > 0$ and $b > 0$.

	Transverse Axis Horizontal	Transverse Axis Vertical
Equation:	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Center:	(h, k)	(h, k)
Foci: (Note: $c^2 = a^2 + b^2$)	$(h+c, k)$ and $(h-c, k)$	$(h, k+c)$ and $(h, k-c)$
Vertices:	$(h+a, k)$ and $(h-a, k)$	$(h, k+a)$ and $(h, k-a)$
Asymptotes:	$y-k = \pm \frac{b}{a}(x-h)$	$y-k = \pm \frac{a}{b}(x-h)$
Graph:		

Example 4: Given $-\frac{(x-2)^2}{9} + (y+3)^2 = 1$, graph the hyperbola and identify the center, vertices, foci and equations of the asymptotes.



$$(y+3)^2 - \frac{(x-2)^2}{9} = 1$$

$$\frac{(y-(-3))^2}{1^2} - \frac{(x-2)^2}{3^2} = 1$$

center: $(2, -3)$

$a = 1$, $b = 3$

$$c^2 = a^2 + b^2 = 1 + 9 = 10$$

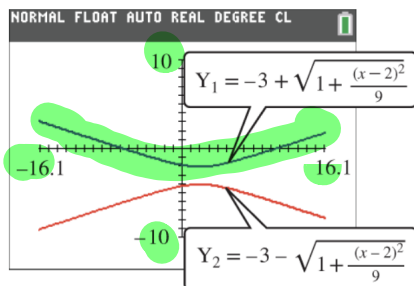
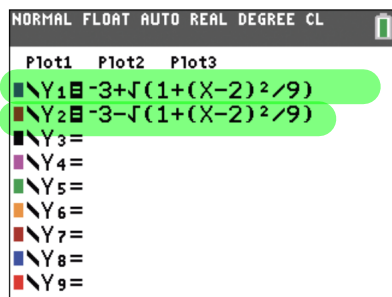
$$c = \sqrt{10}$$

vertex: $(2, -2)$

$$y-k = \pm \frac{a}{b}(x-h)$$

$$y-(-3) = \pm \frac{1}{3}(x-2)$$

Calculator Instructions:



$$(y+3)^2 - \frac{(x-2)^2}{9} = 1$$

$$(y+3)^2 = 1 + \frac{(x-2)^2}{9}$$

$$y+3 = \pm \sqrt{1 + \frac{(x-2)^2}{9}}$$

$$y = -3 \pm \sqrt{1 + \frac{(x-2)^2}{9}}$$

5. 9.1 Continued: Parabolas

In this section we study parabolas from a geometric rather than an algebraic point of view.

We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is the set of all points in the plane that are equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).

