**Day 5 Notes**
*8.1 Sequences and their Notation*

## 1. Sequences

In this section we are going to introduce Sequences and Series which are fundamental mathematical concepts that are important for understanding patterns and are used in practice to model population growth in biology and analyze algorithms in computer science.

**Definition.** Roughly speaking, a **sequence** is an infinite list of numbers with some order or pattern.

**Example 1.** A simple example is the sequence:

first term

$$5, \quad 10, \quad 15, \quad 20, \quad 25, \dots \quad 5n$$
$$5\cdot1 \quad 5\cdot2 \quad 5\cdot3 \quad 5\cdot4, \dots, \dots 5n$$
$$a_1 \quad a_2 \quad a_3 \quad a_4, a_5, \dots, a_n, \dots$$

The numbers in the sequence are called terms and are often written as $a_1$, $a_2$, $a_3$, $\cdots$. The dots mean that the list continues forever.

**Notation:**

$a_2 = 10$ (means the second term of the sequence)

$a_n$ = the general term of the sequence or the nth term of the sequence

$\{a_n\}$ = represents the entire sequence

$\{a_n\} = \{a_1, a_2, a_3, \dots\}$

$\{a_n\} = \{5, 10, 15, \dots\}$

n - is a positive integer
1, 2, 3, 4..

We can describe the pattern of the sequence displayed above by a formula for the general term: $\boxed{a_n = 5n}$

**Example 2.** Find the first three terms and the 100th term of the sequence defined by each formula.

1. $a_n = 2n - 1$

$a_1 = 2\cdot 1 - 1 = 1$

$a_2 = 2\cdot 2 - 1 = 3$

$a_3 = 2\cdot 3 - 1 = 5$

$a_{100} = 2\cdot 100 - 1 = 199$

$\{a_n\} = \{1, 3, 5, \dots\}$

2. $b_n = n^2 - 1$

$b_1 = 1^2 - 1 = 0$

$b_2 = 2^2 - 1 = 3$

$b_3 = 3^2 - 1 = 8$

$b_{100} = 100^2 - 1 = 9999$

$\{b_n\} = \{0, 3, 8, \dots\}$

3. $c_n = \dfrac{n}{n+1}$

$c_1 = \frac{1}{1+1} = 1/2$

$c_2 = \frac{2}{2+1} = \frac{2}{3}$, $c_3 = \frac{3}{3+1} = \frac{3}{4}$

$c_{100} = \frac{100}{100+1} = \frac{100}{101}$

$\{c_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

4. $d_n = \dfrac{(-1)^{n+1}}{n}$

$d_4 = \frac{(-1)^{4+1}}{4} = -1/4$

$d_1 = \frac{(-1)^{1+1}}{1} = \frac{(-1)^2}{1} = 1$

$d_2 = \frac{(-1)^{2+1}}{2} = (-1)^3/2 = -\frac{1}{2}$

$d_3 = \frac{(-1)^{3+1}}{3} = \frac{(-1)^4}{3} = \frac{1}{3}$

$d_{100} = \frac{(-1)^{100+1}}{100} = -\frac{1}{100}$

$\{d_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\}$

**Example 3.** Find the general term of a sequence whose first several terms are given.

(a) $\dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \cdots$

$n=1$, $a_1 = \frac{2}{3} = \frac{1+1}{1+2}$

$n=2$, $a_2 = \frac{3}{4} = \frac{2+1}{2+2}$

$n=3$, $a_3 = \frac{4}{5} = \frac{3+1}{3+2}$

$n=4$, $a_5 = \frac{5}{6} = \frac{4+1}{4+2}$

$n$ $a_n = ?$ $a_n = \frac{n+1}{n+2}$

(b) $-2, 4, -8, 16, -32, \cdots$

$n=1$ $b_1 = -2 = 2^1 (-1)$

$n=2$ $b_2 = 4 = 2^2 (-1)^2$

$n=3$ $b_3 = -8 = 2^3 (-1)^3$

$n=4$ $b_4 = 16 = 2^4 (-1)^4$

$n=5$ $b_5 = -32 = 2^5 (-1)^5$

$n$ $b_n = ?$

$\boxed{b_n = (-1)^n 2^n}$

**Definition.** An **alternating sequence** is a sequence with terms that alternate in sign.

$-, +, -, +, -, +, \ldots \rightarrow$ use $(-1)^n$ in the nth term formula

$+, -, +, -, +, -, \ldots \rightarrow$ use $(-1)^{n+1}$ in the nth term formula

## 2. Recursively defined sequences

Some sequences do not have simple defining formulas like those of the preceding example.

**Definition.** When the nth term of a sequence may depend on some or all of the terms preceding it we call the sequence recursive sequence.

**Example 4.** Example of recursive sequence:

$\boxed{a_1 = 4}$ and $\boxed{a_n = 2a_{n-1} + 1}$ for $n > 1$.

next term

twice the previous term plus 1

$a_1, a_2, \ldots a_{n-1}, a_n, \ldots$

$4$

$4, 2\cdot 4+1, \ldots$

$4, 9, 2\cdot 9+1, \ldots$

$4, 9, 19, \ldots$

$a_1 = 4$

$a_2 = 2a_1 + 1 = 9$

$a_3 = 2\cdot a_2 + 1 = 2\cdot 9 + 1 = 19$

$\vdots$

$a_n = 2a_{n-1}+1$

$a_2 = 2a_1+1$

($n = 2$ in formula)

$a_2 = 2\cdot 4 + 1$

**Example 5.** The Fibonacci Sequence:

Find the first 7 terms of the sequence defined recursively by

$a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$.

next term, previous term + term before previous term

$a_1 = 1$

$a_2 = 1$

$a_3 = a_2 + a_1 = 1+1 = 2$

$a_4 = a_3 + a_2 = 2+1 = 3$

$a_5 = a_4 + a_3 = 3+2 = 5.$

$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$

Factorials form the basis for important series like Taylor series expansions.

**Definition.** The factorial of a positive integer $n$, written as $n!$, is the product of all positive integers less than or equal to n.

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$5! = 120$

$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3628800$

calc: press 10 → MATH → PRB → !

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$

By definition, $0! = 1$

**Example 6.** Evaluate the following expressions.

(a) $\dfrac{8!}{3! \cdot 5!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6 \cdot 7 \cdot 8}{3! \; \cancel{5!}}$ $= \dfrac{6 \cdot 7 \cdot 8}{3!} = \dfrac{\cancel{6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{3}}$ $= 56$

(b) $\dfrac{n!}{(n+2)!} = \dfrac{\cancel{n!}}{\cancel{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} \cdot (n+1)(n+2)}$ $= \dfrac{1}{(n+1)(n+2)}$

**Example 7.** Given the sequence defined by $b_n = \dfrac{n^2}{(n+1)!}$ find $b_1$ and $b_6$.

$b_1 = \dfrac{1^2}{(1+1)!} = \dfrac{1}{2!} = \dfrac{1}{1 \cdot 2} = \dfrac{1}{2}$

$b_6 = \dfrac{6^2}{(6+1)!} = \dfrac{36}{7!} = \dfrac{36}{5040} = \dfrac{1}{140}$

## 4 Series, partial sums and Sigma notation

If we add the terms of a sequence we obtain a **series**.

**Sequence:** $\{a_n\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$

ex: $\{a_n\} = \{5, 10, 15, \ldots, 5n, \ldots\}$

**Series:** $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ (infinite sum of the terms of a sequence)

ex: $5 + 10 + 15 + \ldots$

**Partial sum:** is a finite series or a finite sum (a sum of numbers that add up to a total)

ex: $a_1 + a_2 + a_3 = S_3$ ($S_3$ is called the third partial sum)

ex: $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$
$S_5 = 5 + 10 + 15 + 20 + 25 = 75$

we say $S_5 = 75$

ex: $S_n$ = nth partial sum: sum of the first n terms of a sequence

$\boxed{S_n = a_1 + a_2 + a_3 + \ldots + a_n}$

**Sigma notation:** [1]

is just a short way (compact way) of writing sums using the greek letter to denote sum.

$\sum$ sigma

$S_{10} = a_1 + a_2 + a_3 + \ldots + a_{10} = \sum_{k=1}^{10} a_k$

ex: $5 + 10 + 15 + \ldots = \sum_{i=1}^{\infty} 5i$

$= \sum_{n=1}^{\infty} 5n$

[1] We use **Sigma notation** for both infinite series and finite series.

**Example 1.** Decide if we are working with a finite sum or a series. Then write the sum using summation notation.

1. $5 + 10 + 15 + 20 + 25 + 30$ (sum stops - finite sum)

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$$\boxed{\sum_{i=1}^{6} 5i = S_6}$$

2. $\frac{\sqrt{3}}{1} + \frac{\sqrt{4}}{2} + \frac{\sqrt{5}}{3} + \cdots + \frac{\sqrt{n+2}}{n}$ finite sum

$a_1 \quad a_2 \quad a_3 \quad a_n$

$$S_n = \sum_{i=1}^{n} \frac{\sqrt{i+2}}{i}$$

3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ series or infinite sum

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \ldots$

general term $a_n = \frac{1}{2^n}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

**Example 2.** Write the terms for the sum and evaluate the sum.
$\sum_{n=1}^{5} 2n + 3$ partial sum or finite sum

$$\sum_{n=1}^{5} 2n+3 = \underset{n=1}{2\cdot 1+3} + \underset{n=1}{2\cdot 2+3} + \underset{n=3}{2\cdot 3+3} + \underset{n=4}{2\cdot 4+3} + \underset{n=5}{2\cdot 5+3}$$

$$= 45$$