In this section, you will study three additional categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as $\sin(ku)$ and $\cos(ku)$. Double angles
2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$. $\sin^4 x$, $\tan^6 x$
3. The third category involves *functions of half-angles* such as $\sin\left(\frac{u}{2}\right)$.

$\tan\left(\frac{\theta}{2}\right)$

1 Double-Angle Formulas

The double-angle formulas allow us to express trigonometric functions of twice an angle in terms of functions of the original angle.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$
$= 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

Example 1 Solving a Multiple-Angle Equation

Solve $2\cos x + \sin(2x) = 0$.

$\sin(2\theta) = 2\sin\theta\cos\theta$

used $\sin(2x) = 2\sin x\cos x$

Solution:

$2\cos x + 2\cdot\sin x\cdot\cos x = 0$ $\quad 1 \div 2$

$\cos x + \sin x\cos x = 0$

$\cos x\,(\underbrace{1+\sin x}_{=0}) = 0$

$\cos x = 0 \qquad\qquad 1 + \sin x = 0$

$x = \frac{\pi}{2} + 2n\pi \qquad\qquad \sin x = -1$

$x = \frac{3\pi}{2} + 2n\pi \qquad\qquad x = \frac{3\pi}{2} + 2n\pi$

solution: $x = \frac{\pi}{2} + 2n\pi$

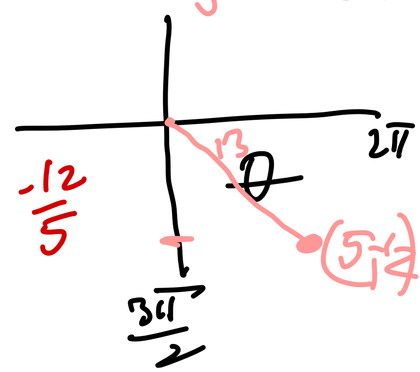
$x = \frac{3\pi}{2} + 2n\pi$

Example 2 Evaluating Functions Involving Double Angles

Use the following to find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

$$\boxed{\cos\theta = \frac{5}{13},} \quad \boxed{\frac{3\pi}{2} < \theta < 2\pi}$$

$y^2 + 5^2 = 13^2$



Solution:

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-12}{13}}{\frac{5}{13}} = \frac{-12}{5}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\sin\theta\left(\frac{5}{13}\right)$

$= 2\left(\frac{-12}{13}\right)\frac{5}{13}$

$= \frac{-120}{169}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{5}{13}\right)^2 = 1$

$\sin^2\theta = 1 - \frac{25}{169}$

$\sin^2\theta = \frac{169-25}{169}$

$\sin^2\theta = \frac{144}{169}$

$\sin\theta = \pm\frac{12}{13}$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{5}{13}\right)^2 - \left(\frac{-12}{13}\right)^2$

$= \frac{25-144}{169} = \frac{-119}{169}$

$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(-12/5)}{1-\left(\frac{-12}{5}\right)^2} = \frac{-264}{25}$

2 Power-Reducing Formulas

The power-reducing formulas express powers of sine and cosine in terms of cosines of multiple angles.

$\cos(2\theta) = 1 - 2\sin^2\theta$

solve for $\sin^2\theta$

$\cos(2\theta) - 1 = -2\sin^2\theta$ $\div(-2)$

$\frac{\cos(2\theta)-1}{-2} = \sin^2\theta$

$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$\int \sin^2\theta \, d\theta = ?$

$\int \frac{1-\cos(2\theta)}{2} d\theta =$ yes...

Example 4 Reducing a Power

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

Solution:

$\boxed{\sin^4 x} = \sin^2 x \cdot \sin^2 x =$

$= \frac{(1-\cos 2x)}{2} \cdot \frac{(1-\cos 2x)}{2} = \frac{1 - \cos 2x - \cos 2x + \cos^2(2x)}{4}$

$= \frac{1 - 2\cos(2x) + \cos^2(2x)}{4} = \frac{1 - 2\cos(2x) + \frac{1+\cos 4x}{2}}{4}$

$= \frac{2 - 4\cos(2x) + 1 + \cos(4x)}{8} = \boxed{\frac{3 - 4\cos(2x) + \cos(4x)}{8}}$

$2(2x)$

$\cos^2(\star) = \frac{1+\cos(2\star)}{2}$

$\cos^2(2x) = \frac{1+\cos(4x)}{2}$

3 Half-Angle Formulas

$\cancel{2}\cdot\frac{\theta}{\cancel{2}} = \theta$

The half-angle formulas express trigonometric functions of half an angle in terms of functions of the original angle.

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$

use $\theta/2$
Formula hold for any angle

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

The sign in the first two formulas depends on the quadrant in which $\frac{\theta}{2}$ lies.

Example 5 Using a Half-Angle Formula

Find the exact value of $\sin 105°$.

Solution:

$\theta = 210°$

$$\sin(105°) = \sin\left(\frac{210°}{2}\right) = \sqrt{\frac{1-\cos(210°)}{2}}$$

$$= \sqrt{\frac{1-(-\sqrt{3}/2)}{2}} = \sqrt{\frac{2\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

Example 6 Solving a Trigonometric Equation

Find all solutions of $1 + \cos^2 x = 2\cos^2\left(\frac{x}{2}\right)$ in the interval $[0, 2\pi)$.

Solution:

$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$

$$1+\cos^2 x = \cancel{2}\left(\frac{1+\cos x}{\cancel{2}}\right)$$

$$1+\cos^2 x = 1+\cos x$$

-1

$$\cos^2 x = \cos x$$

$-\cos x$

$$\cos^2 x - \cos x = 0$$