# Tuesday, April 22nd

We have one more type of asymptote for rational functions.

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be a rational function. If the degree of the top polynomial $P(x)$ is exactly one more than the degree of the bottom polynomial $Q(x)$, then the function $f(x)$ will have a slant or oblique asymptote of the form $y = mx + b$.

**Example:** How to get the slant asymptote. Let $f(x) = \dfrac{x^3}{2x^2 - 2}$.

$$\begin{array}{r|l} & \tfrac{1}{2}x \\ \hline 2x^2-2 & x^3+0x^2+0x+0 \\ & x^3 \quad\quad\quad -x \\ \hline & 0+0+\boxed{x} \end{array}$$

remainder

$$\frac{x^3}{2x^2} = \frac{1}{2}x$$

$$\frac{1}{\cancel{2}}x \cdot \cancel{2}x^2 = x^3$$

$$\frac{1}{2}x(-2) = -x$$

$$\frac{x^3}{2x^2-2} = \frac{1}{2}x + \frac{x}{2x^2-2}$$

remaindr

equation of slant asymptote

$y = mx + b$

$\boxed{y = \frac{1}{2}x}$ line w/ slope $1/2$ & y-intercept 0.

Find the asymptotes and holes(if any) of the rational function

$$f(x) = \frac{(x^2-1)(2x+3)}{(x+1)(x^2+5x+6)} = \frac{(x-1)\cancel{(x+1)}(2x+3)}{\cancel{(x+1)}(x+2)(x+3)}$$

Holes: yes at $x=-1$

$x+1=0$
$x=-1$

V.A: $f(x) = \frac{(x-1)(2x+3)}{(x+2)(x+3)}$

$x=-2$
$x=-3$ are vertical asymptotes

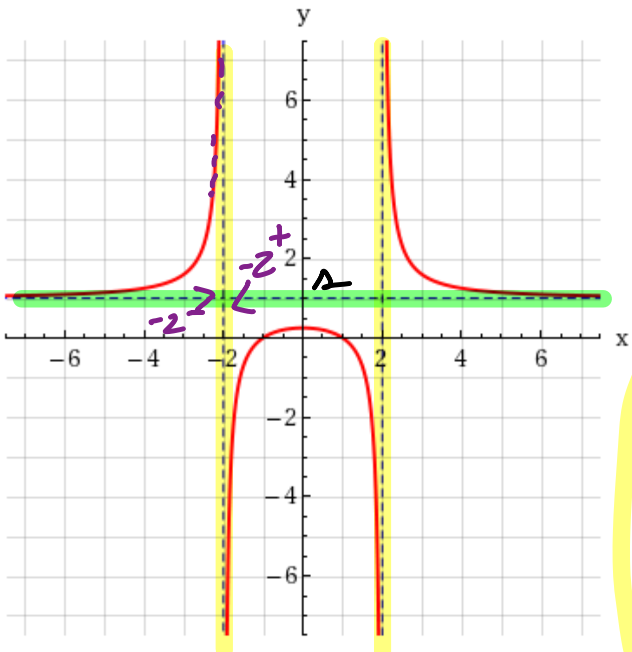
H.A: $f(x) = \frac{(x-1)(2x+3)}{(x+2)(x+3)}$ $\frac{\text{degree } 2}{\text{degree } 2}$

$y = \frac{2}{1}$

$\boxed{\text{H.A. } y=2}$

## 4. Graphs of Rational functions

The graphs of rational functions are more complicated due to the asymptotes. Lets examine the graph of the rational function below.




- no holes
- V.A. $x = -2$
  $x = 2$
- H.A $y = 1$ (end behavior)

$\begin{cases} x \to \infty \\ y \to 1 \end{cases}$ $\begin{cases} x \to -\infty \\ y \to 1 \end{cases}$

limits @ $-\infty$, $\infty$

$\begin{cases} x \to -2^- \\ x \to +\infty \end{cases}$ (opproaches −2 from left)

$\begin{cases} x \to -2^+ \\ x \to -\infty \end{cases}$ (opproaches −2 from right)

limit as $x \to -2$

**How to graph Rational functions**

Graph the rational function $f(x) = \dfrac{x^2 - 2x - 15}{x^2 + 4x}$.

$f(0)=$ not def.
no y intercept

Step 1 Factor

$$\boxed{f(x) = \frac{(x-5)(x+3)}{x(x+4)}} = 0$$

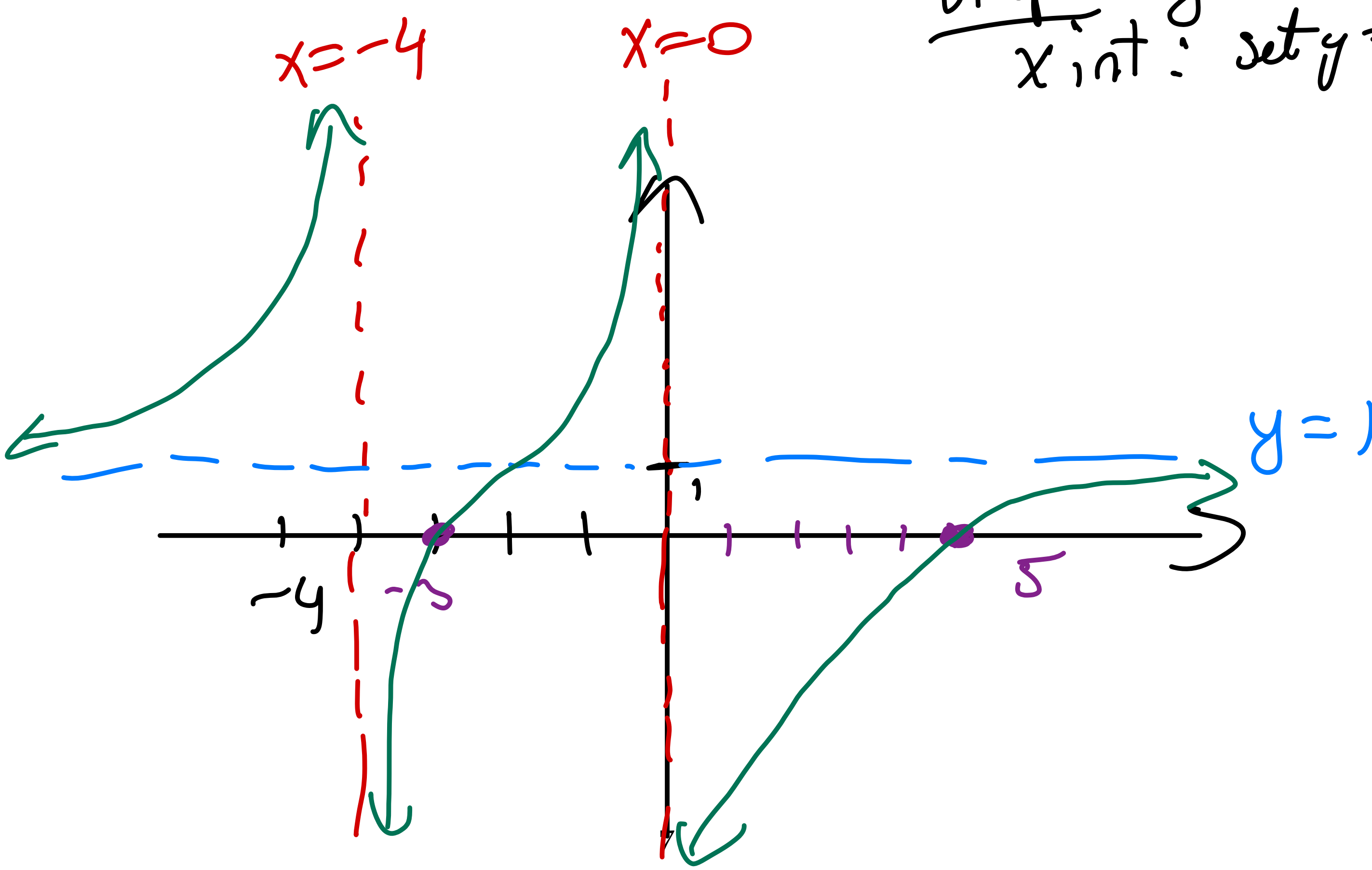
Step 2 holes? NO

Step 3 V.A. $x=-4$
$x=0$

$x+4=0$
$\;\;-4\;\;\;-4$
$x=-4$

Step 4 H.A.: $y = 1/1 \Rightarrow \boxed{y=1}$

Step 5 y int: none
x int: set $y=0$



$$\frac{(x-5)(x+3)}{x(x+4)} = 0$$

$$\boxed{x=5}$$

$$\boxed{x=-3}$$

**Practice:** Graph the rational function $f(x) = \dfrac{x^2 - 4x - 12}{x^2 - x - 6}$

Step 1: Factor $\quad f(x) = \dfrac{\cancel{(x+2)}(x-6)}{(x-3)\cancel{(x+2)}}$

Step 2 Holes? $\quad x = -2$

$f(-2) = \dfrac{-2-6}{-2-3} = 8.5$

$(-2, 1.6)$

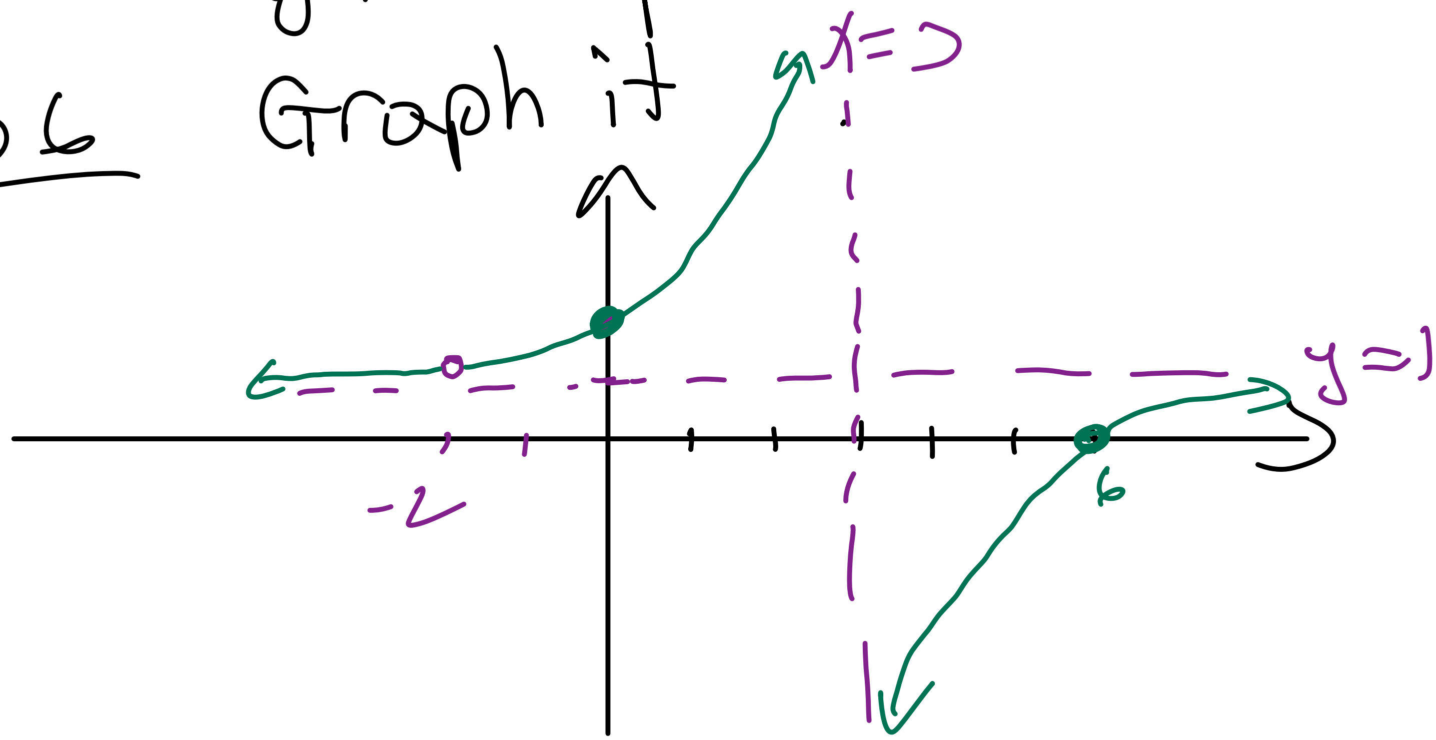
Step 3 V.A.: $x = 3$

Step 4 H.A: $y = 1$

Step 5 x intercept(s): $x = 6$

y intercept: $f(0) = \dfrac{-12}{-6} = 2$

Step 6 Graph it



**Recall:** Composition of Functions:

$$(f\circ g)(x) = f(g(x)) = 2g(x) = \boxed{2\sqrt{x+1}}$$

$f(x) = 2x$

$g(x) = \sqrt{x+1}$

$$(g\circ f)(x) = g(f(x)) = \sqrt{f(x)+1} = \boxed{\sqrt{2x+1}}$$

## 1. One-to-one Functions

A function $f$ is **one-to-one** if there is a one-to-one correspondence between the inputs and the outputs. In other words, $f$ never takes on the same values twice.

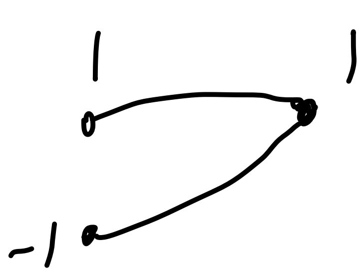
Examples: Is $f$ one-to-one?

$f(x) = x^2$

$f(1) = 1$

$f(-1) = 1$

not one-to-one

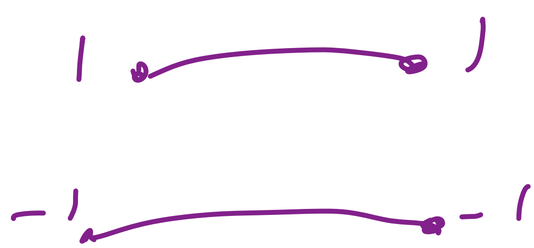



$f(x) = x^3$

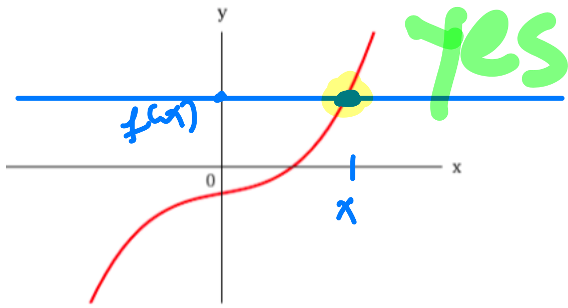
yes one-to-one

$f(1) = 1$

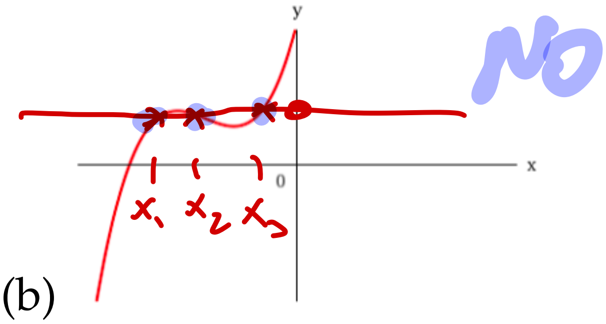
$f(-1) = (-1)^3 = -1$



**How to determine if a function $f$ is one-to-one - Graphically**
Determine if the functions $f$ and $g$ are one-to-one.



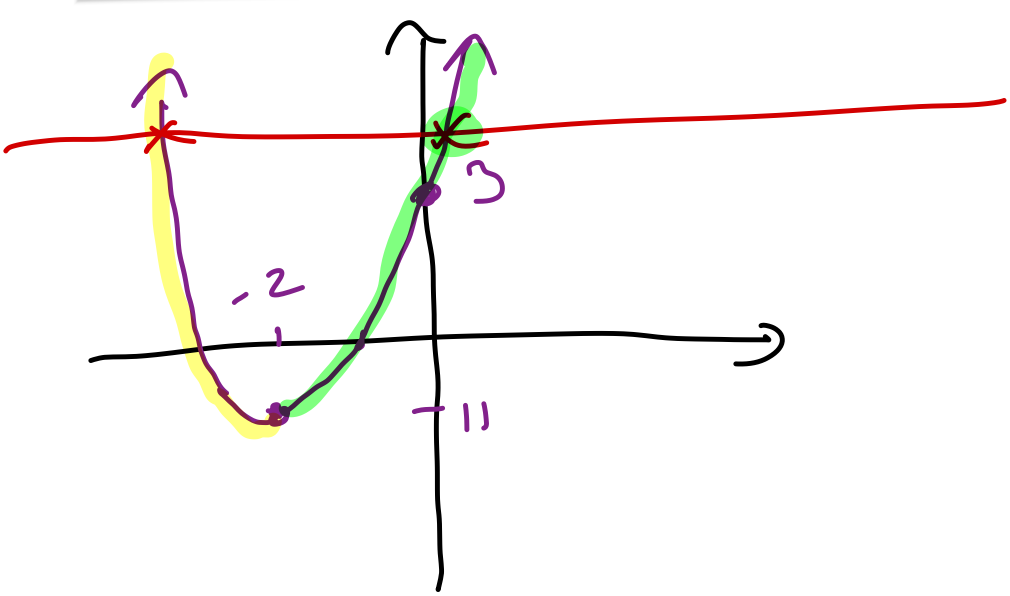

(a)




(b)

**Horizontal Line Test:** If you draw a horizontal line through your function graph and it intersects the graph in only one spot, then your function is one-to-one.

**Example - Restricted Domain**
Graph the function $f(x) = (x+2)^2 - 1$ and decide if it is one-to-one on its entire domain.




Not one-to-one!
fails the
Horizontal line test

If we restrict the domain to $[-2, \infty)$ or $(-\infty, -2]$, then the function becomes one-to-one.

## 2. Inverse Functions

A one-to-one function with domain $A$ and range $B$ has an **inverse function** which takes on as domain the range of $f$ and gives as output the domain of $f$. In other words the inverse of a function $f$ undoes what $f$ does. The inverse, denoted as $f^{-1}$ takes as input the output of $f$ and gives as output the input of $f$, i.e. if $f(x) = y$ then $f^{-1}(y) = x$.

For example if $f$ is the function that takes the input $x$ multiplies it by 5, adds 2 then takes the 5th power of the result, what would the inverse $f^{-1}$ have to be to undo what $f$ did?

$$f(x) = (5x+2)^5$$

$$f^{-1}(x) = \frac{\sqrt[5]{x} - 2}{5}$$

!

**The inverse function $f^{-1}$ reverses the effect of $f$!**

### INVERSE FUNCTION PROPERTY

Let $f$ be a one-to-one function with domain $A$ and range $B$. The inverse function $f^{-1}$ satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function $f^{-1}$ satisfying these equations is the inverse of $f$.

**Practice:** Are $f$ and $g$ inverses of each other?

$$f(x) = 8 - 4x, \quad g(x) = \frac{8-x}{4} = f^{-1}(x)$$

$$(f \circ g)(x) = f(g(x)) = 8 - 4g(x) = 8 - \cancel{4} \cdot \frac{8-x}{\cancel{4}} = 8 - 8 + x = x$$

$$(g \circ f)(x) = g(f(x)) = \frac{8 - f(x)}{4} = \frac{8 - (8-4x)}{4} = \frac{\cancel{8} - \cancel{8} + 4x}{4} = \frac{4x}{4} = x$$

**NOTE:** The function and its inverse switch domain and range:
Domain of $f$ = Rang $f^{-1}$
Range of $f$ = Domain $f^{-1}$

## 3. How to find the inverse of a one-to-one function - Algebraically

Let $f$ be a one-to one function. Then $f(x) = y$ for any $x$ in the domain of $f$ and $f$ has an inverse denoted $f^{-1}$.

**Example:** Let $\boxed{f(x) = \frac{x-9}{x+9}}$. *

(a) Graph $f$. Is $f$ one-to-one?

(b) Find the domain of $f$.

(c) Can you find the range of $f$ without graphing it?

(d) Find the inverse of $f$.
**Steps to find inverse functions:**

Step1 Write $y = \frac{x-9}{x+9}$

Step2 Switch $x$ and $y$: $\boxed{\frac{x}{1} = \frac{y-9}{y+9}}$

Step3 Solve for $y$: $x(y+9) = y - 9$

$-y \qquad -y$

$y = \ldots$

$x(y+9) - y = -9$

$xy + 9x - y = -9$

(e) Can you find the range of $f$ now using the information you have about $f^{-1}$?

$-9x \qquad -9x$

$xy - y = -9 - 9x$

$\frac{y \cdot 2}{2} = \frac{5}{2}$

$\frac{y\cdot(x-1)}{x-1} = \frac{-9-9x}{x-1}$

$\boxed{y = \frac{-9-9x}{x-1}}$

this is $f^{-1}$

$f^{-1}(x) = \frac{-9-9x}{x-1}$