Day 3 Notes
9.4 Series and their Notation

1. Series, partial sums and Sigma notation

If we add the terms of a sequence we obtain a **series**.

Sequence: $\{a_n\} = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$

ex: $\{a_n\} = \{5, 10, 15, \ldots, 5n, \ldots\}$

Series: $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ (infinite sum of the term of a sequence)

ex: $5 + 10 + 15 + \ldots$

Partial sum: is a finite series or a finite sum (a sum of numbers that add up to a total)

ex: $a_1 + a_2 + a_3 = S_3$ (S_3 is called the third partial sum)

ex: $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$
$S_5 = 5 + 10 + 15 + 20 + 25 = 75$

we say $\boxed{S_5 = 75}$

ex: S_n = nth partial sum: sum of the first n terms of a sequence $\boxed{S_n = a_1 + a_2 + a_3 + \ldots + a_n}$

Sigma notation: [1]

is just a short way (compact way) of writing sums using the greek letter to denote sum.

Σ sigma

$$S_{10} = a_1 + a_2 + a_3 + \ldots + a_{10} = \sum_{k=1}^{10} a_k$$

ex: $5 + 10 + 15 + \ldots = \sum_{i=1}^{\infty} 5i$

$= \sum_{n=1}^{\infty} 5n$

[1] We use **Sigma notation** for both infinite series and finite series.

Example 1. Decide if we are working with a finite sum or a series. Then write the sum using summation notation.

1. $5 + 10 + 15 + 20 + 25 + 30$ (sum stops - finite sum)

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

$$\boxed{\sum_{i=1}^{6} 5i = S_6}$$

2. $\dfrac{\sqrt{3}}{1} + \dfrac{\sqrt{4}}{2} + \dfrac{\sqrt{5}}{3} + \cdots + \dfrac{\sqrt{n+2}}{n}$ finite sum

$a_1 \quad a_2 \quad a_3 \quad a_n$

$$S_n = \sum_{i=1}^{n} \frac{\sqrt{i+2}}{i}$$

3. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ series or infinite sum

$a_1 \quad a_2 \quad a_3 \quad a_4 \ldots$

general term $a_n = \dfrac{1}{2^n}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Example 2. Write the terms for the sum and evaluate the sum.
$\sum_{n=1}^{5} 2n + 3$ partial sum or finite sum

$$\sum_{n=1}^{5} 2n+3 = \underset{n=1}{2\cdot 1+3} + \underset{n=1}{2\cdot 2+3} + \underset{n=3}{2\cdot 3+3} + \underset{n=4}{2\cdot 4+3} + \underset{n=5}{2\cdot 5+3}$$

$$= 45$$

PARTIAL SUMS OF AN ARITHMETIC SEQUENCE $S_n = a_1 + a_2 + \cdots + a_n$

For the arithmetic sequence given by $a_n = a + (n-1)d$, the **nth partial sum**

$$S_n = \underset{a_1}{a} + \underset{a_2}{(a+d)} + \underset{a_3}{(a+2d)} + \underset{a_4}{(a+3d)} + \cdots + \underset{a_n}{[a+(n-1)d]}$$

is given by either of the following formulas.

1. $S_n = \frac{n}{2}[2a + (n-1)d]$ **2.** $\boxed{S_n = n\left(\frac{a+a_n}{2}\right)}$

$$\boxed{S_n = n\left(\frac{a_1+a_n}{2}\right)} \quad (*)$$

Example 4: a) Find the sum of the first 100 numbers.

$$\underset{a_1}{1} + \underset{a_2}{2} + \underset{a_3}{3} + 4 + \cdots + \underset{a_{100}}{100}$$

$1, 2, 3, \ldots, 100$

$a = 1$

$d = 1$

$S_{100} = ?$

$$\boxed{S_{100} = 100\left(\frac{1+100}{2}\right)} = 100 \cdot \frac{101}{2} = 5050$$

$(*)$ w/ $n = 100$

b) Find the sum of the first 50 odd numbers.

$$\underset{a_1}{1} + \underset{a_2}{3} + \underset{a_3}{5} + \underset{a_4}{7} + \underset{a_5}{9} + \cdots + \boxed{99} = ?$$

$a_{50} = ?$

$1, 3, 5, 7, 9, \ldots$

$a = 1$

$d = 2$

$a_n = a + (n-1)d$

Find $a_{50} = ?$

$a_{50} = a + (50-1)d$

$= 1 + 49 \cdot 2 \Rightarrow a_{50} = 99$

$$\boxed{S_{50} = 50\left(\frac{1+99}{2}\right) = 2500}$$

c) Sum the first 25 terms of the sequence $3, -1, -5, -9, \cdots$

$$3, \underbrace{\ }_{-4} -1, \underbrace{\ }_{-4} -5, \underbrace{\ }_{-4} -9, \ldots$$

check if it's arithmetic

$a = 3$

$d = -4$

want $S_{25} = ?$

$$\underset{a_1}{3} + \underset{a_2}{(-1)} + \underset{a_3}{(-5)} + \underset{a_4}{(-9)} + \cdots + \underset{a_{25}}{\boxed{-93}}$$

$a_n = a + (n-1)d$

$\boxed{a_n = 3 + (n-1)(-4)}$

$a_{25} = 3 + (25-1)(-4) = -93$

$$\boxed{S_{25} = 25\left(\frac{3+(-93)}{2}\right) = -1125}$$

3. Partial Sums of Geometric Sequences

The partial sum s_n of a geometric sequence is given by the formula below.

PARTIAL SUMS OF A GEOMETRIC SEQUENCE

$a_1, a_2, a_3, \ldots, a_n, \ldots$

$a_n = ar^{n-1}$

For the geometric sequence defined by $a_n = ar^{n-1}$, the ***n*th partial sum**

$$S_n = \underbrace{a}_{a_1} + \underbrace{ar}_{a_2} + \underbrace{ar^2}_{a_3} + \underbrace{ar^3}_{a_4} + \underbrace{ar^4}_{a_5} + \cdots + \underbrace{ar^{n-1}}_{a_n} \qquad r \neq 1$$

is given by

$$\boxed{S_n = a\frac{1-r^n}{1-r}}$$

Example 7: Find the following partial sum of a geometric sequence:

$\underbrace{1}_{a_1} + \underbrace{4}_{a_2} + \underbrace{16}_{a_3} + \cdots + \underbrace{4096}_{a_7}$

add the first 7 term

$$\boxed{S_7 = 1\,\frac{1-4^7}{1-4} = \frac{1-4^7}{-3} = 5461}$$

$$S_n = a\,\frac{1-r^n}{1-r}$$

$1, 4, 16, 64, 4096, \ldots$

$a_n = ar^{n-1}$

$a = 1$

$r = 4$

$a_n = 1 \cdot 4^{n-1} \Rightarrow a_n = 4^{n-1}$

$4096 = 4^{n-1}$

$4^6 = 4^{n-1} \Rightarrow 6 = n-1$

$n = 7$

4. Infinite Geometric Series

An infinite geometric series is a series of the form

$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$

SUM OF AN INFINITE GEOMETRIC SERIES

If $|r| < 1$, then the infinite geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges and has the sum

$$S = \frac{a}{1-r}$$

If $|r| \geq 1$, the series diverges.

$|r| < 1$

$\boxed{-1 < r < 1}$

Example 8: Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$

(b) $1 + \frac{7}{5} + \frac{49}{25} + \cdots$

a) $\frac{2}{5^0} + \frac{2}{5^1} + \frac{2}{25} + \frac{2}{125} + \cdots = \sum_{k=1}^{\infty} \frac{2}{5^k}$

$25 \to 5^2$, $125 \to 5^3$

$2 \cdot \frac{1}{5} = \frac{2}{5}$

$\frac{2}{5} \cdot \frac{1}{5} = \frac{2}{25}$

$a = 2$

$r = \frac{1}{5}$

$|r| = \left|\frac{1}{5}\right| < 1$ series converges to the value $\boxed{S = \frac{a}{1-r}} = \frac{2}{1-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = 2.5$

b) $1, \frac{7}{5}, \frac{49}{25}, \ldots$ geom sequence

$1 \cdot \frac{7}{5} = \frac{7}{5}$

$\frac{7}{5} \cdot \frac{7}{5} = \frac{49}{25} \cdots$

$\boxed{\begin{array}{l} a = 1 \\ r = \frac{7}{5} \end{array}}$

$r = \frac{7}{5}$

$\left|\frac{7}{5}\right| \geq 1$

$\frac{7}{5} = 1.4$

so series diverges.