1. Exponential Functions

Exponential and logarithmic functions represent two fundamental mathematical relationships that are inverses of each other. Exponential functions, written in the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$, model growth and decay phenomena where a quantity changes at a rate proportional to its current value. These functions appear in numerous real-world applications, from compound interest and population growth to radioactive decay.

An exponential function is a function of the form $f(x) = \_\_\_\_$, where the base b $\_\_\_\_$ and b $\_\_\_\_$
Note: The independent variable x is in the $\_\_\_\_\_\_\_\_\_\_\_\_$!

Ex:

The exponential function with base e is $f(x) = \_\_\_\_$, where $e = \left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. Note $e \approx \_\_\_\_$.

Calculator buttons:

Recall rules of exponents:

$a^m \cdot a^n = a^{m+n}$	$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\frac{a^m}{a^n} = a^{m-n}$	$(ab)^n = a^n b^n$	$a^{-n} = \frac{1}{a^n}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$

1. Section 3.1 Exponential Functions

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An exponential function is a function of the form $f(x) =$ b^x, where the base b >0 and b $\neq 1$ $(1^x = 1)$
Note: The independent variable x is in the exponent !

Ex: $f(x) = 3^x$

$f(x) = (1/4)^x$

Euler constant

The exponential function with base e is $f(x) =$ e^x, where $\boxed{e = \left(1 + \frac{1}{n}\right)^n}$ as $n \to \infty$. Note $e \approx$ 2.718.

Calculator buttons:
1) Exponent $\boxed{\wedge}$
2) e : $\boxed{\text{2nd}}$ $\boxed{\div}$
3) e^x : $\boxed{\text{2nd}}$ $\boxed{\ln}$

Recall rules of exponents:

$a^m \cdot a^n = a^{m+n}$	$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\frac{a^m}{a^n} = a^{m-n}$	$(ab)^n = a^n b^n$	$\boxed{a^{-n} = \frac{1}{a^n}}$ *	$a^{\frac{1}{n}} = \sqrt[n]{a}$

$a^{-2} = \left(\frac{1}{a}\right)^2 = \frac{1^2}{a^2} = \frac{1}{a^2}$

$\sqrt{a^1} = a^{1/2}$

$\sqrt[3]{a^2} = a^{2/3}$

Example 1. Use a calculator(if needed) to evaluate the function at the indicated value of x. Round to three decimal places.

a. $f(x) = 5^x$ at $x = 2$

$f(2) = 5^2 = 25$

b. $f(x) = 2^x$ at $x = -3$

$f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

c. $f(x) = \left(\frac{1}{4}\right)^x$ at $x = 0$

$f(0) = (1/4)^0 = 1$

d. $f(x) = 2.3^x$ at $x = -1$

$f(-1) = (2.3)^{-1} = \frac{1}{2.3^1} = \frac{1}{2.3}$

$= 0.435$

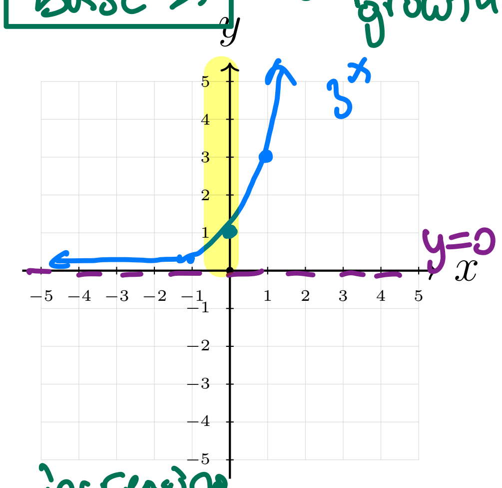
Example 2. Graph the function and write the domain and range in interval notation.

$\boxed{0 < b < 1}$

1) $f(x) = 3^x$

x	$y = 3^x$	
-1	$3^{-1} = \frac{1}{3} = 0.\bar{3}$	$(-1, .\bar{3})$
0	$3^0 = 1$	$(0,1)$
1	$3^1 = 3$	$(1,3)$

$\boxed{\text{base} > 1}$ exp growth



H.A. $y = 0$

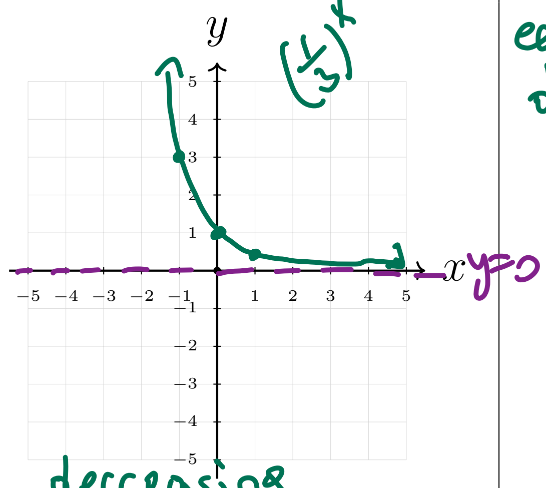
domain: $(-\infty, \infty)$

range: $(0, \infty)$

2) $f(x) = \left(\frac{1}{3}\right)^x$

x	$y = \left(\frac{1}{3}\right)^x$	
-1	$\left(\frac{1}{3}\right)^{-1} = \left(\frac{3}{1}\right)^1 = 3$	$(-1,3)$
0	$(1/3)^0 = 1$	$(0,1)$
1	$(1/3)^1 = \frac{1}{3} = 0.\bar{3}$	$(1, 1/3)$

exp. decay



H.A. $y = 0$

domain $(-\infty, \infty)$

range : $(0, \infty)$

$\cancel{3^{?} = 0}$

Example 3. Use the graph of the parent function to describe the transformation that yields the graph of g.

$g(x) = 2^{x+1} - 3$

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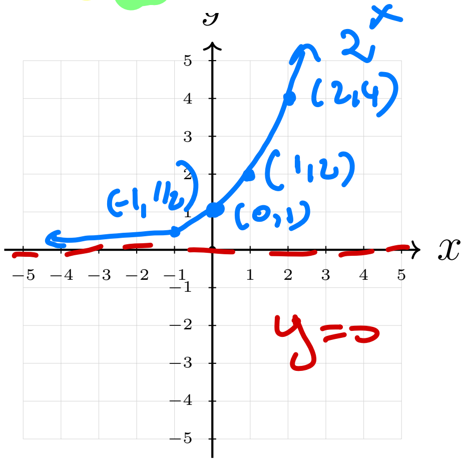
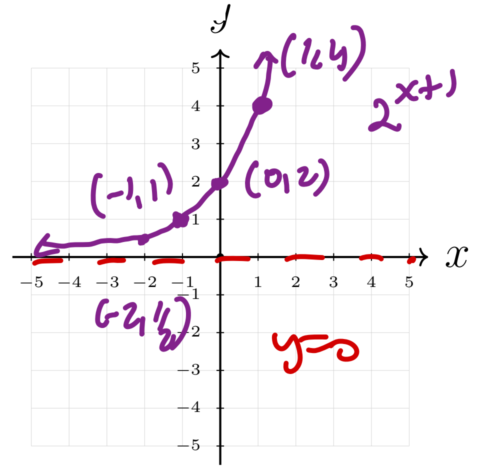
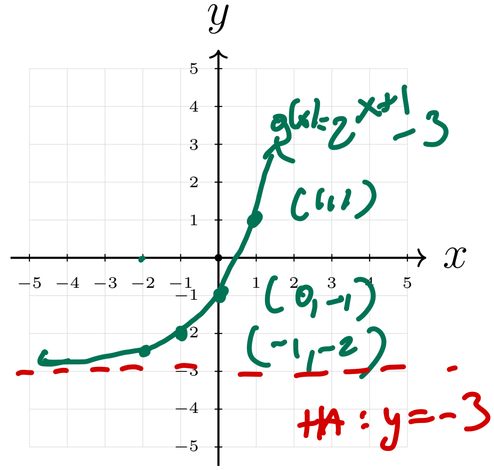






parent function

$f(x)=2^x$

left 1 unit

$(2,4) \to (2-1, 4) = (1,4)$

$(1,2) \to (0,2)$

$(0,1) \to (-1,1)$

$(-1,1/2) \to (-2,1/2)$

down 3 units

$(1,4) \to (1, 4-3) = (1,1)$

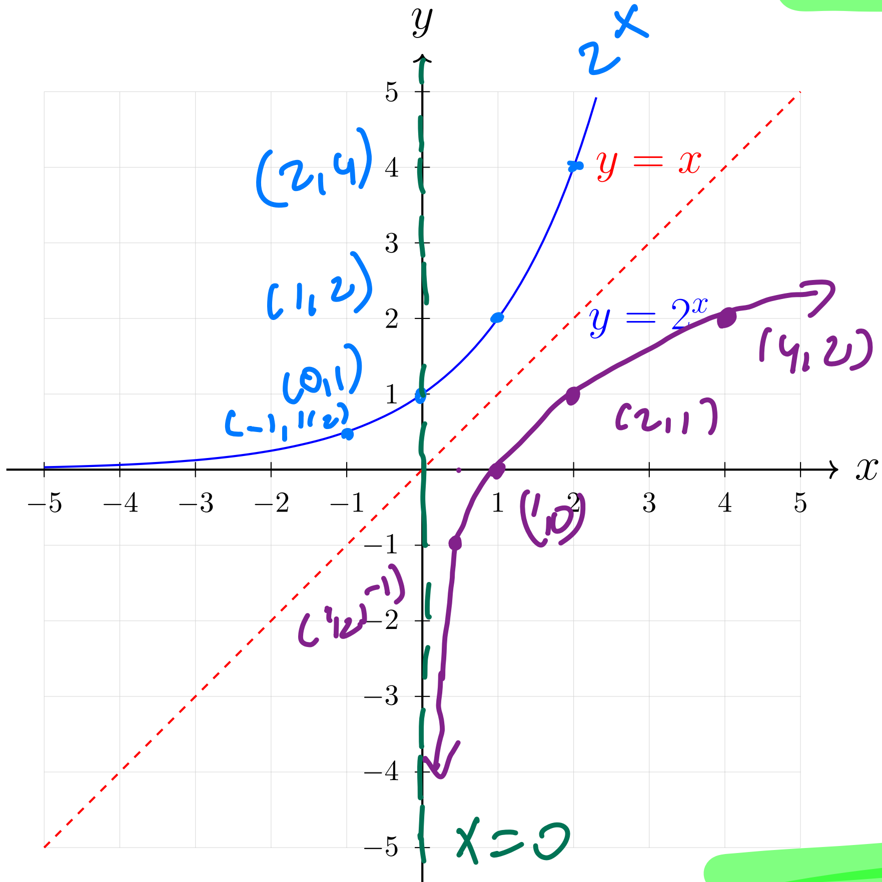
$(0,2) \to (0,-1)$

$(-1,1) \to (-1,-2)$

$(-2,1/2) \to (-2, \underbrace{1/2-3}_{-2.5})$

2. Section 3.2 Logarithmic Functions

The inverse of the exponential function $f(x) = b^x$ is called **logarithmic function with base a** $f(x)=2^x$



Recall: 2^x
- increasing
- Domain $(-\infty,\infty)$
- Range $(0,\infty)$
- HA: $y=0$

Notation for logarithmic function: $f(x)=\log_2 x$

Domain: $(0,\infty)$

Range: $(-\infty,\infty)$

Asymptote: y axis: $x=0$
V.A.

A logarithmic function is a function of the form $f(x) = \underline{\log_b x}$, where $x \underline{>0}$, and the base $b \underline{>0}$ and $b \underline{\neq 1}$.

$f(x) = \log_b(\underbrace{\text{quantity in } x}_{>0})$ domain

Special Logarithms:

1) common log: $b=10$ $\quad f(x)=\log_{10} x = \log x$

2) natural log: $b=e$ $\quad f(x)=\log_e x = \ln x$

Example 4. Write the logarithmic equation in exponential form.

1. $\log_2 8 = 3$
2. $\log_{10} 7 = x$
3. $\log_{1/2} y = 5$
4. $\ln 1 = 0$

1) $\log_2 8 = 3$ $\quad 2^3 = 8$ $\quad \log_2 8 = 3$

2) $\log_{10} 7 = x$ $\quad 10^x = 7$

3) $\log_{1/2} y = 5$ $\quad \left(\frac{1}{2}\right)^5 = y$

4) $\ln 1 = 0$ $\quad \log_e 1 = 0$ $\quad e^0 = 1$

Example 5. Write the exponential form in logarithmic equation.

1. $5^x = 23$
2. $8^{-2} = \dfrac{1}{64}$
3. $b^n = 9$

1) $\log_5 23 = x$

2) $\log_8 \frac{1}{64} = -2$

3) $\log_b 9 = n$

Example 6. Use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.

1. $f(x) = \log_5 x$ at $x = 25$
2. $f(x) = \log_2 x$ at $x = 8$
3. $f(x) = \log x$ at $x = 100000$
4. $f(x) = \ln x$ at $x = e^9$

   $\ln e^9 = 9$
5. $f(x) = \log x$ at $x = 0.001$

   $\log 0.001 = -3$
6. $f(x) = \ln x$ at $x = \frac{1}{e^5}$

1) $\log_5 25 = ?$ $\quad \boxed{5^? = 25}$ $\quad 5^2 = 25$

$\log_5 25 = 2$

2) $2^? = 8$ $\quad 2^3 = 8 \therefore \log_2 8 = 3$

3) $\log_{10} 100{,}000 = ?$

$10^? = 100{,}000 \Rightarrow 10^5 = 100{,}000$

$\log_{10} 100{,}000 = 5$

4) $\log_e e^9 = ?$

$e^? = e^9 \Rightarrow e^9 = e^9$

5) $\log_{10} 0.001 = ?$

$10^? = 0.001$

$10^? = \frac{1}{1000}$ or $10^? = 10^{-3}$

6) $\ln \frac{1}{e^5} = ?$

$\log_e e^{-5} = -5$

$f(1) = \log_4 1 = 0$

7. $f(x) = \log_4 x$ at $x = 1$ $\quad 4^? = 1 \quad 4^0 = 1$

$\log_4 1 = ?$

8. $f(x) = \log_2 x$ at $x = 0$ $\quad 2^? = 0$ DNE

$f(0) = \log_2 0 = ?$

9. $f(x) = \log_2 x$ at $x = -16$ $\quad 2^? = -16$ DNE

$f(-16) = \log_2 -16 = ?$

Example 7. Use a calculator to evaluate the function at the indicated value of x.

$10^? = 123$

1. $f(x) = \log x$ at $x = 123$ $\quad f(123) = \log_{10}(123) = 2.089$

2. $f(x) = \ln x$ at $x = 2$ $\quad f(2) = \ln 2 = 0.693 \quad e^? = 2$

Basic Properties of Logarithms with base b, where $b > 0$, $b \neq 1$, and $x > 0$

1) $\log_b 1 = 0$ because $b^0 = 1$

2) $\log_b b = 1$ because $b^1 = b$

3) $\log_b b^x = x$ because $f(f^{-1}(x)) = x$

4) $b^{\log_b x} = x$ because $f^{-1}(f(x)) = x$

EX Use the properties of logarithms to simplify the expression, without using a calculator.

1) $\log_3 3^{5x} \overset{P_3}{=} 5x$

2) $\log_{\sqrt{2}} 1 \overset{P_1}{=} 0$

$\sqrt{2}^{\,0} = 1$

3) $6 \log_4(1/4) = 6 \cancel{\log_4} 4^{-1} \overset{P_3}{=} 6(-1) = -6 \quad \frac{1}{4} = 4^{-1}$

4) $\cancel{4^{\log_4}}{}^{(5x+y)} = 5x + y$

EX Use the properties of natural logarithms to rewrite the expression, without using a calculator.

1) $\ln e^2 = \log_e e^2 = 2$ 3) $\ln e = \log_e e^1 = 1$ $e^1 = 1$

2) $5 \ln 1 = 5 \cdot 0 = 0$ 4) $e^{\ln(4.2)} = \cancel{e}^{\cancel{\log_e} 4.2} = 4.2$

Example 8. Find the domain in interval notation.

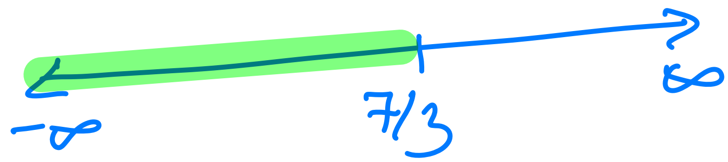
1. $y = \log_{1/2}(7 - 3x) = f(x)$

$7 - 3x > 0$

$-3x > -7 \quad | \div (-3)$

$x < -7/-3$

$x < 7/3$

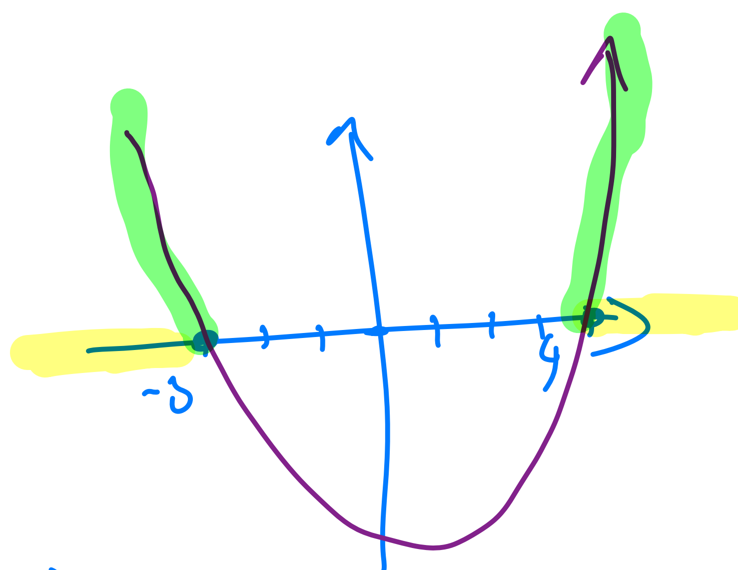


Domain: $(-\infty, 7/3)$

2. $y = \log_2(x^2 - x - 12)$

$x^2 - x - 12 > 0$

$(x+3)(x-4) > 0$



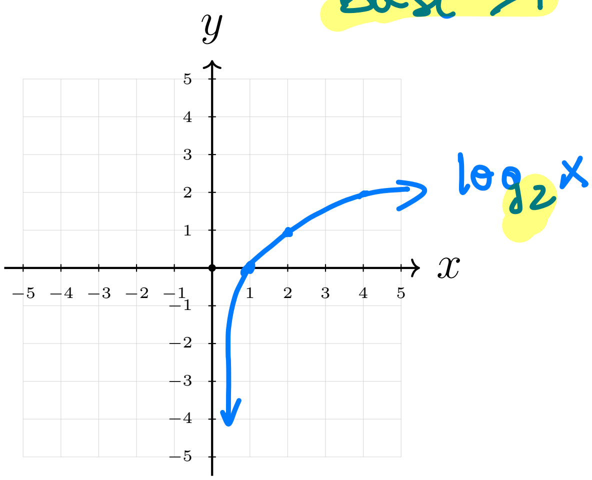
Domain: $(-\infty, -3) \cup (4, \infty)$

$0 < b < 1$

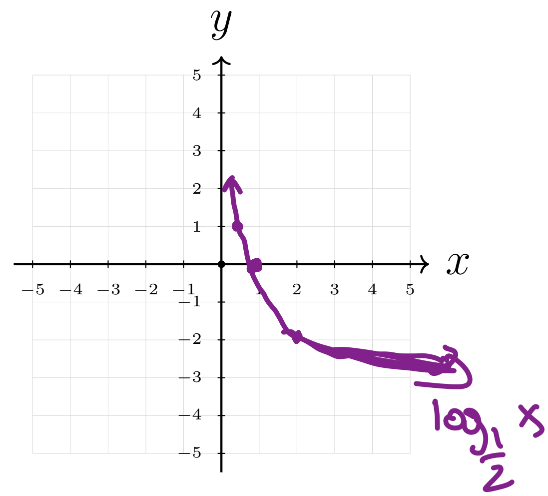
$f(x) = \log_{\frac{1}{2}} x$

Graphs of logarithmic functions

base > 1



increasing



decreasing

VA: $x=0$

$x=1$

$\log_{1/2} 1 = ?$

$\left(\frac{1}{2}\right)^0 = 1$

$\log_{\frac{1}{2}} \frac{1}{2} = 1$

$(1/2, 1)$

$\log_{\frac{1}{2}} 4 = ?$

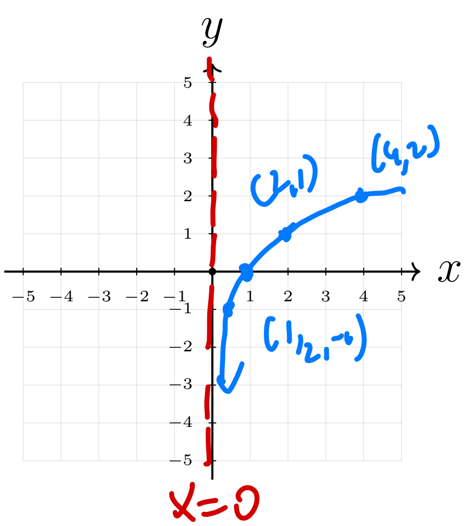
$\left(\frac{1}{2}\right)^? = 4$

$\left(\frac{1}{2}\right)^? = 2^2$

$\left(\frac{1}{2}\right)^{-2} = 2^2$

Example 9. Use the graph of the parent function to describe the transformation that yields the graph of g. Then find the domain and range in interval notation, the vertical asymptote and the x-intercept.

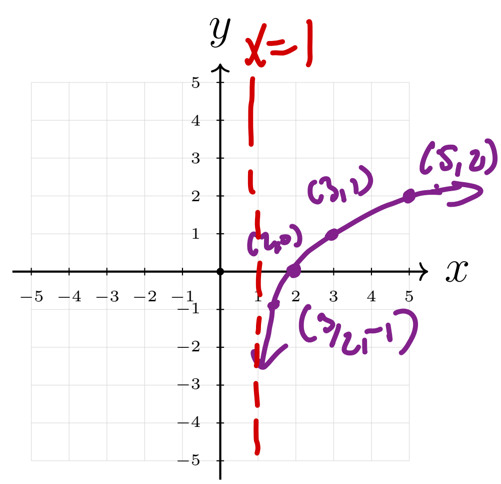
$g(x) = -\log_2(x-1)$



parent function

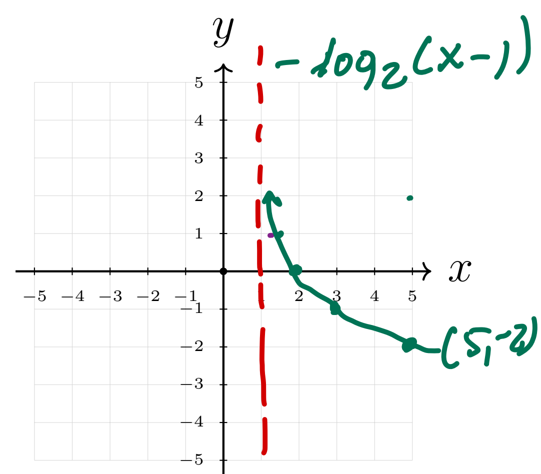
$\log_2 x$

VA: $x=0$



right 1 unit

$\log_2(x-1)$



VA: $x=1$

reflect over x axis