Mth 146: Day 19
11.10 Taylor and Maclaurin Series
(Solutions)

1 Part 1: New Concepts & Worked Examples

Which Functions Have Power Series?

In *Representations of Functions as Power Series* we could only find power series for functions that come from the geometric series. Today we answer two bigger questions:

- **Which functions have power series representations?**
- **How can we find such representations?**

We will see that some of the most important functions in calculus, such as e^x and $\sin x$, can be written as power series.

Suppose f can be represented by a power series

$$\textbf{(1)} \qquad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \qquad |x-a| < R$$

What must the coefficients c_n be? Differentiate term by term and plug in $x = a$ each time:

$$f(x) = c_0 + c_1(x-a) + \cdots \quad \Rightarrow \quad f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + \cdots \quad \Rightarrow \quad f'(a) = c_1$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + \cdots \quad \Rightarrow \quad f''(a) = 2c_2$$

$$f'''(x) = 2\cdot 3c_3 + \cdots \quad \Rightarrow \quad f'''(a) = 3!c_3$$

The pattern: $f^{(n)}(a) = n!\,c_n$. Solving for c_n gives the formula in the box below. (We use the conventions $0! = 1$ and $f^{(0)} = f$.)

Theorem. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

So *if* f has a power series expansion at a, it must be of the form

Taylor series of f at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

This is called the **Taylor series of the function f at a** (or *about* a, or *centered at* a). The special case $a = 0$ comes up so often that it gets its own name:

Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Notes:

- Finding the Taylor series of f **does not guarantee** that the series **adds up to f(x)**. There are functions that are not equal to the sum of their Taylor series.
- A power series representation at a is **unique**: **no matter how we find it, it is the Taylor series.** So every series we found in *Representations of Functions as Power Series* is a Taylor series!

Example 1. We know that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$. Confirm that this is the Maclaurin series of $f(x) = \dfrac{1}{1-x}$ by computing $c_n = \dfrac{f^{(n)}(0)}{n!}$.

Solution: We compute derivatives and evaluate at 0:

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$
$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$$
$$f''(x) = \frac{1 \cdot 2}{(1-x)^3} \qquad f''(0) = 1 \cdot 2$$
$$f'''(x) = \frac{1 \cdot 2 \cdot 3}{(1-x)^4} \qquad f'''(0) = 1 \cdot 2 \cdot 3$$

In general, $f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$, so $f^{(n)}(0) = n!$. Thus

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1 \qquad \text{and} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \ \checkmark$$

Example 2. For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

Solution: If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Radius of convergence: let $a_n = \dfrac{x^n}{n!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$$

so by the Ratio Test the series converges for all x: $\mathbf{R} = \infty$.

When Is a Function Equal to Its Taylor Series?

The partial sums of a Taylor series are polynomials:

The n**th-degree Taylor polynomial of** f **at** a is

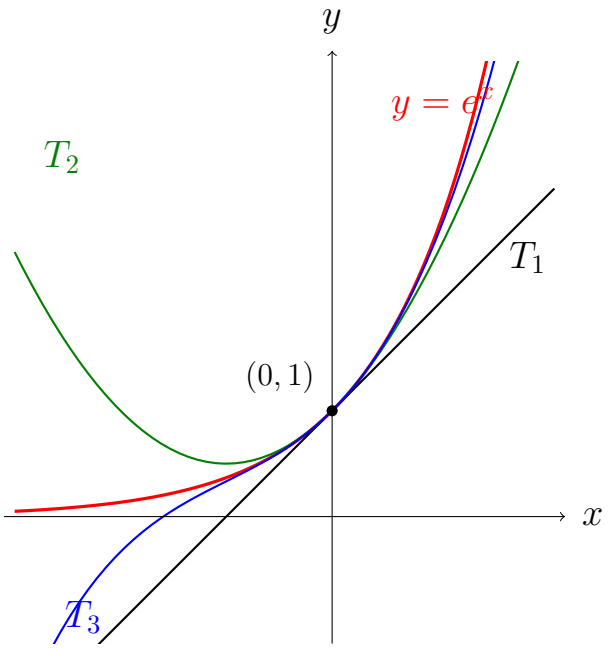
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For $f(x) = e^x$ at $a = 0$, Example 2 gives

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$



$y = e^x$ (red) with its Taylor polynomials T_1, T_2, T_3 at $a = 0$

As n increases, $T_n(x)$ appears to approach e^x. In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

The difference between f and its Taylor polynomial is called the **remainder**:

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x).$$

Theorem. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for } |x - a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

To show $R_n(x) \to 0$ we usually use:

Taylor's Inequality. If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

A useful fact when applying these theorems:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for every real number } x$$

This is true because the series $\sum \dfrac{x^n}{n!}$ converges for all x (Example 2), so its nth term must go to 0.

Example 3. Show that e^x is equal to the sum of its Maclaurin series.

Solution: If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| = e^x \le e^d$. So Taylor's Inequality with $a = 0$ and $M = e^d$ says

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \le d.$$

The same $M = e^d$ works for every n, and

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

By the Squeeze Theorem, $R_n(x) \to 0$ for all x. So e^x equals the sum of its Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

In particular, putting $x = 1$ gives **the number e as an infinite series**:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Example 4. Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution: We have $f^{(n)}(2) = e^2$ for all n, so putting $a = 2$ in the definition of a Taylor series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n.$$

As in Example 2, the Ratio Test gives $R = \infty$, and as in Example 3, $R_n(x) \to 0$. So

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n \qquad \text{for all } x.$$

We now have **two** power series for e^x. The Maclaurin series is better for x near 0, and the Taylor series at $a = 2$ is better for x near 2.

Example 5. Find the Maclaurin series for $\sin x$ and show that it represents $\sin x$ for all x.

Solution: Arrange the computation in two columns:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

The derivatives repeat in a cycle of four, so the Maclaurin series is

$$\begin{aligned} &f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we know $\left|f^{(n+1)}(x)\right| \leq 1$ for all x. Taking $M = 1$ in Taylor's Inequality:

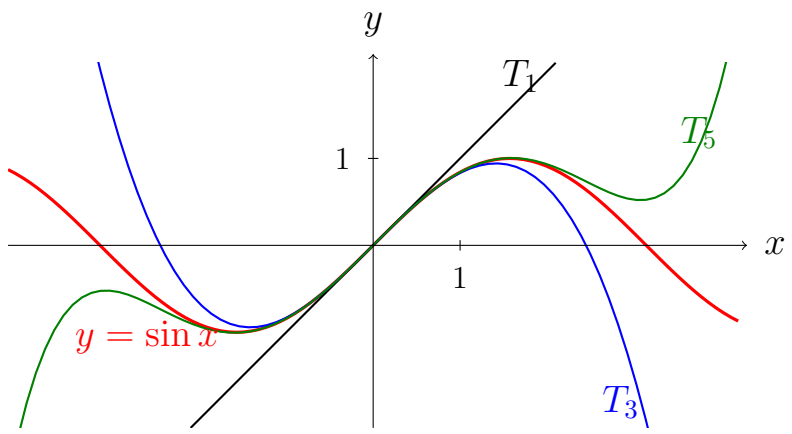
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} \to 0.$$

So $R_n(x) \to 0$ by the Squeeze Theorem, and $\sin x$ equals the sum of its Maclaurin series.

The Maclaurin polynomials of $\sin x$ are

$$T_1(x) = x \qquad T_3(x) = x - \frac{x^3}{3!}$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$



$y = \sin x$ (red) with its Maclaurin polynomials T_1, T_3, T_5

Notice that as n increases, $T_n(x)$ becomes a **better approximation** to $\sin x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

Example 6. Find the Maclaurin series for $\cos x$.

Solution: We could proceed as in Example 5, but it's easier to differentiate the Maclaurin series for $\sin x$ term by term:

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\end{aligned}$$

Differentiating does not change the radius of convergence, so this series also converges for all x.

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x\end{aligned}$$

Fun fact: The Maclaurin series for e^x, $\sin x$, and $\cos x$ were discovered by Newton. They tell us that we know *everything* about each of these functions if we know all of its derivatives at the single number 0.

Example 7. Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\dfrac{\pi}{3}$.

Solution: Arranging our work in columns:

$$\begin{aligned} f(x) &= \sin x & f\left(\tfrac{\pi}{3}\right) &= \tfrac{\sqrt{3}}{2} \\ f'(x) &= \cos x & f'\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{2} \\ f''(x) &= -\sin x & f''\left(\tfrac{\pi}{3}\right) &= -\tfrac{\sqrt{3}}{2} \\ f'''(x) &= -\cos x & f'''\left(\tfrac{\pi}{3}\right) &= -\tfrac{1}{2} \end{aligned}$$

and this pattern repeats. So the Taylor series at $\frac{\pi}{3}$ is

$$\begin{aligned} & f\left(\tfrac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \tfrac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \tfrac{\pi}{3}\right)^2 + \cdots \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!}\left(x - \tfrac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \tfrac{\pi}{3}\right)^2 \\ &\quad - \frac{1}{2 \cdot 3!}\left(x - \tfrac{\pi}{3}\right)^3 + \cdots \end{aligned}$$

(The proof that it represents $\sin x$ for all x is just like Example 5.) In sigma notation, separate the terms that contain $\sqrt{3}$:

$$\begin{aligned} \sin x = & \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \tfrac{\pi}{3}\right)^{2n} \\ & + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \tfrac{\pi}{3}\right)^{2n+1} \end{aligned}$$

Use the Maclaurin series for x near 0 and the Taylor series at $\frac{\pi}{3}$ for x near $\frac{\pi}{3}$.

Example 8. Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution: We start by computing derivatives:

$$\begin{aligned} f(x) &= (1+x)^k & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\ f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \end{aligned}$$

and in general

$$f^{(n)}(0) = k(k-1)\cdots(k-n+1).$$

Therefore the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n$$

By the Ratio Test, $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|k-n|}{n+1}|x| \to |x|$, so the series converges if $|x| < 1$ and diverges if $|x| > 1$: $\mathbf{R = 1}$.

This series is called the **binomial series**. Its coefficients are the **binomial coefficients**:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

The Binomial Series. If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Notes:

- If k is a **positive integer**, the series **stops** (the terms are eventually 0), and we get the ordinary Binomial Theorem.
- Whether the series converges at the endpoints ± 1 depends on k: it converges at 1 if $-1 < k \leq 0$ and at both endpoints if $k \geq 0$.

Example 9. For the function $f(x) = \dfrac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.

Solution: Rewrite $f(x)$ so we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Use the binomial series with $k = -\frac{1}{2}$ and x replaced by $-\frac{x}{4}$:

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\left[1 + \left(-\tfrac{1}{2}\right)\left(-\tfrac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\tfrac{x}{4}\right)^2\right. \\
&\qquad \left. + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\tfrac{x}{4}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{8}x + \frac{1\cdot 3}{2!\,8^2}x^2 + \frac{1\cdot 3\cdot 5}{3!\,8^3}x^3 + \cdots\right]
\end{aligned}$$

The general term inside the brackets is $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\,8^n}x^n$.
This converges when $\left|-\frac{x}{4}\right| < 1$, that is, $|x| < 4$, so $\mathbf{R = 4}$.

Table of Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R = 1$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R = 1$$
$$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$$

New Taylor Series from Old

Because a power series representation is unique, we can build new Taylor series by manipulating the ones in the table instead of computing $f^{(n)}(a)$. Just like in *Representations of Functions as Power Series*, we can:

- replace x by an expression of the form cx^m,
- multiply or divide the series by such an expression,
- differentiate or integrate **term by term**,
- and add, subtract, multiply, or divide Taylor series.

Example 10. Find the Maclaurin series for (a) $f(x) = x\cos x$ and (b) $f(x) = \ln(1+3x^2)$.

Solution: (a) Multiply the Maclaurin series for $\cos x$ by x:

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!} \qquad \text{for all } x$$

(b) Replace x by $3x^2$ in the Maclaurin series for $\ln(1+x)$:

$$\ln(1+3x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n x^{2n}}{n}$$

From the table, this converges for $|3x^2| < 1$, that is, $|x| < \frac{1}{\sqrt{3}}$, so $\mathbf{R = \frac{1}{\sqrt{3}}}$.

Example 11. Find the function represented by the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!}$.

Solution: Write

$$\sum_{n=0}^{\infty}(-1)^n\frac{2^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}.$$

This is the series for e^x with x replaced by $-2x$. So the series represents $\mathbf{e^{-2x}}$.

Example 12. Find the sum of the series

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots.$$

Solution: In sigma notation,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\cdot 2^n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n}.$$

This matches the table entry for $\ln(1+x)$ with $x=\frac{1}{2}$. So

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\cdot 2^n}=\ln\left(1+\tfrac{1}{2}\right)=\ln\mathbf{\tfrac{3}{2}}.$$

2 Part 2: Now You Try It

Use the definition of a Taylor series or the table of important Maclaurin series, as indicated.

1. Use the definition to find the Maclaurin series for $f(x) = e^{5x}$. What is the radius of convergence?

   Solution: $f^{(n)}(x) = 5^n e^{5x}$, so $f^{(n)}(0) = 5^n$ and $c_n = \dfrac{5^n}{n!}$:

   $$e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n.$$

   Ratio Test: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{5|x|}{n+1} \to 0 < 1$ for all x, so $\mathbf{R} = \infty$.

2. Use the definition to find the Taylor series for $f(x) = \ln x$ centered at $a = 2$. What is the radius of convergence?

   Solution: $f(2) = \ln 2$, $f'(x) = \dfrac{1}{x}$, $f''(x) = -\dfrac{1}{x^2}$, $f'''(x) = \dfrac{2}{x^3}$, $f^{(4)}(x) = -\dfrac{3!}{x^4}$, and in general $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$ for $n \geq 1$. So

   $$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n-1}(n-1)!}{2^n\, n!} = \frac{(-1)^{n-1}}{n\, 2^n} \qquad (n \geq 1)$$

   $$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 2^n} (x-2)^n.$$

Ratio Test:
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}\cdot\frac{|x-2|}{2} \to \frac{|x-2|}{2} < 1 \iff |x-2| < 2,$$
so $\mathbf{R = 2}$.

3. Use the definition to find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$.

Solution: $f(\pi) = \cos\pi = -1$, $f'(\pi) = -\sin\pi = 0$, $f''(\pi) = -\cos\pi = 1$, $f'''(\pi) = \sin\pi = 0$, $f^{(4)}(\pi) = \cos\pi = -1$, and the pattern repeats. Only even powers survive:

$$\begin{aligned}\cos x &= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}.\end{aligned}$$

As with $\cos x$ at 0, $\mathbf{R} = \infty$.

4. Use the binomial series to find the Maclaurin series for $f(x) = \dfrac{1}{(1+x)^3}$. What is the radius of convergence?

Solution: Use $k = -3$:

$$\begin{aligned}(1+x)^{-3} &= 1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 \\ &\quad + \frac{(-3)(-4)(-5)}{3!}x^3 + \cdots \\ &= 1 - 3x + 6x^2 - 10x^3 + \cdots\end{aligned}$$

In general
$\binom{-3}{n} = \dfrac{(-1)^n\, 3 \cdot 4 \cdots (n+2)}{n!} = \dfrac{(-1)^n (n+1)(n+2)}{2}$, so

$$\frac{1}{(1+x)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n, \qquad \mathbf{R = 1.}$$

3 Part 3: Reflection

For each item below, rate your comfort level and jot a quick note: What clicked? What's still fuzzy? What would help?

1. **Use the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$ to find a Taylor or Maclaurin series.** 1 2 3 4 5

2. **Write the nth-degree Taylor polynomial $T_n(x)$ and explain how it approximates f.** 1 2 3 4 5

3. **Recall the important Maclaurin series (e^x, $\sin x$, $\cos x$, $\tan^{-1} x$, $\ln(1+x)$, $(1+x)^k$) and their radii of convergence.** 1 2 3 4 5

4. **Build new series from the table by substituting, multiplying, differentiating, or integrating.** 1 2 3 4 5

5. **Recognize a series as a known function to find its sum.** 1 2 3 4 5