MTH 146 – Calculus II
Practice Exam 2
ANSWER KEY

Name: _____ Date: _____

Show all of your work. Whenever you use a test or theorem, name it and verify that its conditions are satisfied. Answers without supporting work will not receive full credit.

1. Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-2}^{3} \frac{9}{x^4}\,dx$$

The integrand is discontinuous at $x = 0$, which lies inside $[-2, 3]$, so we split:

$$\int_{-2}^{3} \frac{9}{x^4}\,dx = \int_{-2}^{0} \frac{9}{x^4}\,dx + \int_{0}^{3} \frac{9}{x^4}\,dx.$$

$$\int_{-2}^{0} \frac{9}{x^4}\,dx = \lim_{t\to 0^-} \left[-\frac{3}{x^3}\right]_{-2}^{t} = \lim_{t\to 0^-} \left(-\frac{3}{t^3} - \frac{3}{8}\right) = \infty$$

since $t^3 \to 0^-$. Because one piece diverges, the integral **diverges**. (Note: ignoring the discontinuity and plugging in the endpoints gives a wrong finite answer.)

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{(5n-1)!}{(5n+1)!}$$

$(5n+1)! = (5n+1)(5n)(5n-1)!$, so

$$a_n = \frac{(5n-1)!}{(5n+1)(5n)(5n-1)!} = \frac{1}{(5n+1)(5n)} \to 0.$$

The sequence **converges to** 0.

3. Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = n^2 e^{-5n}$$

Write $a_n = \dfrac{n^2}{e^{5n}}$, which has the form $\dfrac{\infty}{\infty}$. Using the function $f(x) = \dfrac{x^2}{e^{5x}}$ and l'Hôpital's Rule twice:

$$\lim_{x\to\infty} \frac{x^2}{e^{5x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2x}{5e^{5x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2}{25e^{5x}} = 0.$$

The sequence **converges to** 0.

4. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty}\left(\frac{9}{e^n}+\frac{2}{n(n+1)}\right)$$

First part: $\displaystyle\sum_{n=1}^{\infty}\frac{9}{e^n}$ is geometric with $a=\frac{9}{e}$ and $r=\frac{1}{e}$; $|r|<1$, so

$$\sum_{n=1}^{\infty}\frac{9}{e^n}=\frac{9/e}{1-1/e}=\frac{9}{e-1}.$$

Second part: by partial fractions, $\dfrac{2}{n(n+1)}=\dfrac{2}{n}-\dfrac{2}{n+1}$, so the partial sums telescope:

$$s_n=\left(2-\tfrac{2}{2}\right)+\left(\tfrac{2}{2}-\tfrac{2}{3}\right)+\cdots+\left(\tfrac{2}{n}-\tfrac{2}{n+1}\right)=2-\frac{2}{n+1}\to 2.$$

Both series converge, so the sum converges to

$$\frac{9}{e-1}+2=\boxed{\frac{2e+7}{e-1}}.$$

For Problems 5–14: Determine whether the series is *absolutely convergent*, *conditionally convergent*, or *divergent*. **State which test you are using**, verify its conditions, and justify your answer. Choose from the following tests:

- Test for Divergence
- Geometric Series
- p-Series
- Integral Test
- Direct Comparison Test
- Limit Comparison Test
- Alternating Series Test
- Ratio Test
- Root Test

5.

$$\sum_{n=1}^{\infty}\frac{5}{\sqrt{n^2+7}}$$

Positive terms. Limit Comparison Test with $b_n=\dfrac{1}{n}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{5n}{\sqrt{n^2+7}}=\lim_{n\to\infty}\frac{5}{\sqrt{1+7/n^2}}=5>0.$$

Since $\sum\frac{1}{n}$ is the divergent harmonic series ($p=1$), the series **diverges**.

6.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+9}$$

Alternating Series Test with $b_n=\dfrac{n^2}{n^3+9}>0$:

- $\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty}\dfrac{1/n}{1+9/n^3}=0.$
- Decreasing: for $f(x)=\dfrac{x^2}{x^3+9}$, $f'(x)=\dfrac{2x(x^3+9)-3x^4}{(x^3+9)^2}=\dfrac{x(18-x^3)}{(x^3+9)^2}<0$ when $x>\sqrt[3]{18}\approx 2.62$, so $\{b_n\}$ is decreasing for $n\geq 3$.

So the series converges.

Absolute convergence: $\sum|a_n|=\sum\dfrac{n^2}{n^3+9}$. Limit Comparison with $\frac{1}{n}$: $\lim\dfrac{n^3}{n^3+9}=1>0$, and $\sum\frac{1}{n}$ diverges, so $\sum|a_n|$ diverges.

The series is **conditionally convergent**.

7.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+10n+26}$$

$f(x)=\dfrac{1}{x^2+10x+26}=\dfrac{1}{(x+5)^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies:

$$\int_1^{\infty}\frac{dx}{(x+5)^2+1}=\lim_{t\to\infty}\Big[\arctan(x+5)\Big]_1^t=\lim_{t\to\infty}\big(\arctan(t+5)-\arctan 6\big)=\frac{\pi}{2}-\arctan 6.$$

The integral is finite, so the series converges. All terms are positive, so it is **absolutely convergent**.

(Also valid: Direct Comparison, $\frac{1}{n^2+10n+26}<\frac{1}{n^2}$, a convergent p-series with $p=2$.)

8.

$$\sum_{n=1}^{\infty}\frac{1+\sin n}{e^n}$$

Since $0\leq 1+\sin n\leq 2$, we have $0\leq\dfrac{1+\sin n}{e^n}\leq\dfrac{2}{e^n}$ for all $n\geq 1$. $\sum\frac{2}{e^n}=2\sum\left(\frac{1}{e}\right)^n$ is a convergent geometric series ($|r|=\frac{1}{e}<1$). By the Direct Comparison Test the series converges; its terms are nonnegative, so it is **absolutely convergent**.

9.

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!}$$

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-4)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-4)^n}\right| = \lim_{n\to\infty} \frac{4}{(2n+3)(2n+2)} = 0 < 1.$$

The series is **absolutely convergent**.

10.

$$\sum_{n=1}^{\infty} \frac{1}{5+e^{-n}}$$

$$\lim_{n\to\infty} \frac{1}{5+e^{-n}} = \frac{1}{5+0} = \frac{1}{5} \neq 0,$$

so the series **diverges** by the Test for Divergence.

11.

$$\sum_{n=2}^{\infty} \frac{n+7}{(n+3)^3}$$

Positive terms. Limit Comparison Test with $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+7)}{(n+3)^3} = \lim_{n\to\infty} \frac{1+7/n}{(1+3/n)^3} = 1 > 0.$$

$\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), so the series converges; it has positive terms, so it is **absolutely convergent**.

12.

$$\sum_{n=5}^{\infty} 2\left(1+\frac{1}{n}\right)^{n^2}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 2^{1/n}\left(1+\frac{1}{n}\right)^n = 1 \cdot e = e > 1,$$

so the series **diverges**. (Also: $a_n \to \infty$, so the Test for Divergence works too.)

13.

$$\sum_{n=1}^{\infty} \frac{e^{2/n}}{n}$$

Since $\frac{2}{n} > 0$, $e^{2/n} > 1$, so $\dfrac{e^{2/n}}{n} > \dfrac{1}{n}$ for all $n \geq 1$. $\sum \frac{1}{n}$ is the divergent harmonic series, so the series **diverges** by the Direct Comparison Test. (Limit Comparison with $\frac{1}{n}$ also works: the limit is $e^0 = 1$.)

14.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{2^{n+1}(n+1)^3}\cdot\frac{2^n n^3}{3^n} = \frac{3}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = \frac{3}{2} > 1,$$

so the series **diverges**.

15. Approximate the sum of the series correct to four decimal places.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^9}$$

$b_n = \dfrac{1}{n^9}$ is positive, decreasing, and $\to 0$, so the Alternating Series Estimation Theorem gives $|s - s_n| \le b_{n+1}$. We need $b_{n+1} < 0.00005$:

$$b_3 = \frac{1}{3^9} \approx 0.0000508 \text{ (too big)}, \qquad b_4 = \frac{1}{4^9} \approx 0.0000038 < 0.00005.$$

So s_3 is enough:

$$s_3 = 1 - \frac{1}{2^9} + \frac{1}{3^9} = 1 - 0.0019531 + 0.0000508 \approx 0.998098.$$

The sum is $s \approx \mathbf{0.9981}$.

16. Find the radius of convergence and the interval of convergence of the series.

$$\sum_{n=2}^{\infty}\frac{(x+2)^n}{2^n \ln n}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\frac{2^n \ln n}{(x+2)^n}\right| = \frac{|x+2|}{2}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{|x+2|}{2},$$

since by l'Hôpital $\lim_{x\to\infty}\dfrac{\ln x}{\ln(x+1)} = \lim_{x\to\infty}\dfrac{x+1}{x} = 1$. Converges when $\frac{|x+2|}{2} < 1 \iff |x+2| < 2 \iff -4 < x < 0$, so $\mathbf{R = 2}$.

Endpoints:

- $x = -4$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test ($\frac{1}{\ln n}$ decreases to 0).
- $x = 0$: $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by Direct Comparison with $\sum \frac{1}{n}$, since $\ln n < n \Rightarrow \frac{1}{\ln n} > \frac{1}{n}$.

$\mathbf{I} = [-\mathbf{4},\ \mathbf{0})$.

17. Find the radius of convergence and the interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 8^n}{\sqrt{n}}\, x^n$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{8^{n+1}x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{8^n x^n} \right| = 8|x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 8|x|.$$

Converges when $8|x| < 1 \iff |x| < \frac{1}{8}$, so $\mathbf{R} = \mathbf{\frac{1}{8}}$.

Endpoints:

- $x = \frac{1}{8}$: $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test ($\frac{1}{\sqrt{n}}$ decreases to 0).
- $x = -\frac{1}{8}$: $\sum \dfrac{(-1)^n(-1)^n}{\sqrt{n}} = \sum \dfrac{1}{\sqrt{n}}$, a divergent p-series ($p = \frac{1}{2} \le 1$).

$\mathbf{I} = \left(-\mathbf{\frac{1}{8}},\ \mathbf{\frac{1}{8}}\right]$.

18. Find a power series representation for the function centered at $x = 0$, and find its radius of convergence.

$$f(x) = \frac{1}{(8+x)^2}$$

Start from a geometric series:

$$\frac{1}{8+x} = \frac{1}{8} \cdot \frac{1}{1-\left(-\frac{x}{8}\right)} = \frac{1}{8} \sum_{n=0}^{\infty} \left(-\frac{x}{8}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{8^{n+1}}, \qquad \left|\frac{x}{8}\right| < 1.$$

Notice $\dfrac{d}{dx}\left[\dfrac{1}{8+x}\right] = -\dfrac{1}{(8+x)^2}$, so

$$\frac{1}{(8+x)^2} = -\frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{8^{n+1}} = -\sum_{n=1}^{\infty} \frac{(-1)^n\, n\, x^{n-1}}{8^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n\, x^{n-1}}{8^{n+1}}.$$

Reindexing ($n \to n+1$):

$$\boxed{\frac{1}{(8+x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)\, x^n}{8^{n+2}}}$$

Differentiation does not change the radius, so $\mathbf{R} = \mathbf{8}$.

19. Find a power series representation for the function centered at $x = 0$, and find its radius of convergence.

$$f(x) = \ln(11 - x)$$

Notice $\dfrac{d}{dx}\ln(11-x) = -\dfrac{1}{11-x}$, and

$$\frac{1}{11-x} = \frac{1}{11}\cdot\frac{1}{1-\frac{x}{11}} = \sum_{n=0}^{\infty}\frac{x^n}{11^{n+1}}, \qquad \left|\frac{x}{11}\right| < 1.$$

Integrate term by term:

$$\ln(11-x) = -\int\frac{dx}{11-x} = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)11^{n+1}} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,11^n}.$$

At $x=0$: $\ln 11 = C$. So

$$\boxed{\ln(11-x) = \ln 11 - \sum_{n=1}^{\infty}\frac{x^n}{n\,11^n}}$$

Integration does not change the radius, so $\mathbf{R = 11}$.

20. Find a power series representation for the function centered at $x=0$, and determine the interval of convergence.

$$f(x) = \frac{x^2}{x^4+81}$$

$$\frac{x^2}{x^4+81} = \frac{x^2}{81}\cdot\frac{1}{1-\left(-\frac{x^4}{81}\right)} = \frac{x^2}{81}\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{81^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{3^{4n+4}}}$$

This converges when $\left|\frac{x^4}{81}\right| < 1 \iff |x| < 3$, so $R = 3$. At $x = \pm 3$ the geometric ratio has absolute value 1, so the series diverges. $\mathbf{I} = (\mathbf{-3}, \mathbf{3})$.