Day 8 Notes
Chapter 3 Functions and Relations

1. Functions and Relations

In nearly every physical phenomenon, we observe that one quantity depends on another. For example, the cost of tuition depends on the number of credits you take, or the area of a circle depends on its radius.

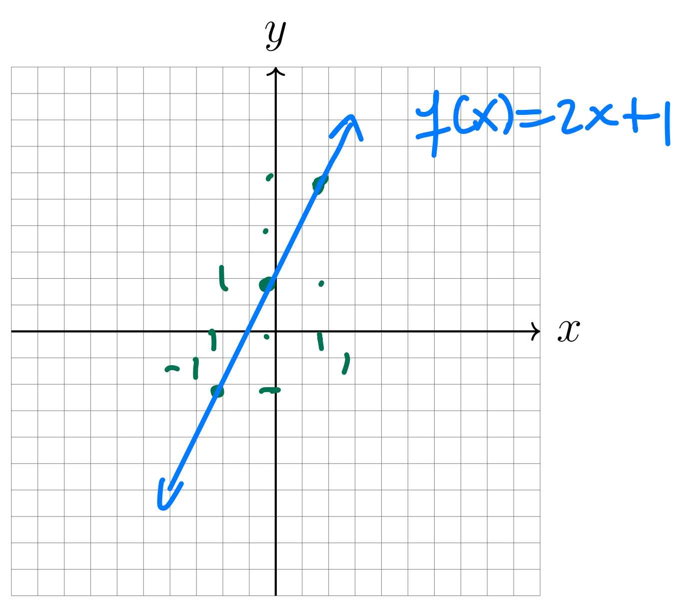
We use the term **function** to describe this dependence of one quantity on another.
We will use letters such as $f, g, h \cdots$ to represent functions.
We can represent a function in **four different ways**. For example, we can use the letter f to represent a rule that doubles the input and then add 1.

1. **Verbally:** f is the rule that doubles the input x and then adds 1.
2. **Algebraically**: $y = 2x + 1$
   $f(x) = 2x + 1$

   x: input or independent variable
   y: output or dependent variable
3. **Numerically:**

x	-1	0	1	2
y	-1	1	3	5

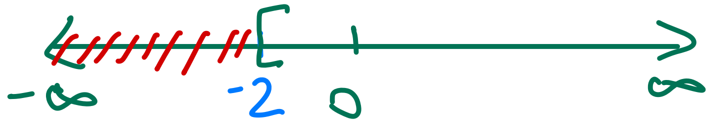
   a table
4. **Graphically**:



Formal definition of a function:
A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.
Set A is called the domain of the function and it consists of all valid inputs.
Set B is called the range of the function and it consists of all possible outputs $f(x)$ as x varies throughout the domain.

Example 1. (Interval notation for Domains and Range) Find the domain and range for $f(x) = \sqrt{x+2}$

$\sqrt{0} = 0$
$\sqrt{-1}$ = not real #
$\sqrt{-2+2} = \sqrt{0} = 0$



Domain: $[-2, \infty)$

Range: $[0, \infty)$

Example 2. Identify if the following relations are functions. If yes, identify the domain and range.

1. $y = x^2 + 1$

$x = -1$
$y = (-1)^2 + 1 = 2$

yes

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

2.

x	y
1	2
1	3
2	4

1 has two output

NO

3.

x	y
1	5
2	7
3	5

yes

Domain: $\{1, 2, 3\}$
Range: $\{5, 7\}$

2. Piecewise defined functions

Sometimes a function has different rules for different portions of the domain and when that happens we call those **piecewise defined functions**.

$$f(x) = \begin{cases} \text{formula 1, piece 1 of domain} \\ \text{formula 2, piece 2} \\ \text{formula 3, piece 3} \end{cases}$$

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 3, & x > 0 \end{cases}$$

Example 3. So, for example, if a cell phone company charges $40 a month if you use less than 500 minutes, but for every minute beyond the allowed 500 minutes you get charged a quarter a minute, the function that calculates the cost of the phone plan will be a piecewise defined function.
Find a formula for the cost of this phone plan, $C(x)$ where x is the number of minutes used in a month period.

$$C(x) = \begin{cases} 40, & x < 500 \\ 40 + 0.25(x-500), & x \geq 500 \end{cases}$$

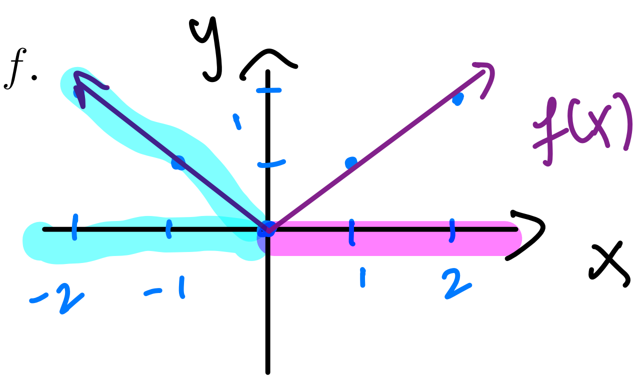
$x = 510$

$40 + 10(0.25)$

$40 + 0.25(510-500) = 40 + .25(10) = 42.5$

Example 4. Recall the absolute value function $f(x) = |x|$.

1. Graph f.



$f(x) = |x|$

$f(2) = |2| = 2$

$f(-3) = |-3| = 3$

2. Rewrite f as a piecewise defined function.

$$f(x) = \begin{cases} -1x + 0, & x \le 0 \\ +x + 0, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$

3. Evaluating a Function

Evaluating a function at a specific input value means plug in that specific value in the formula to get your output.

Example 5. Let $f(x) = 2x^2 + 3x + 1$. Evaluate

1. $f(2) = 2(2)^2 + 3(2) + 1 = 15$

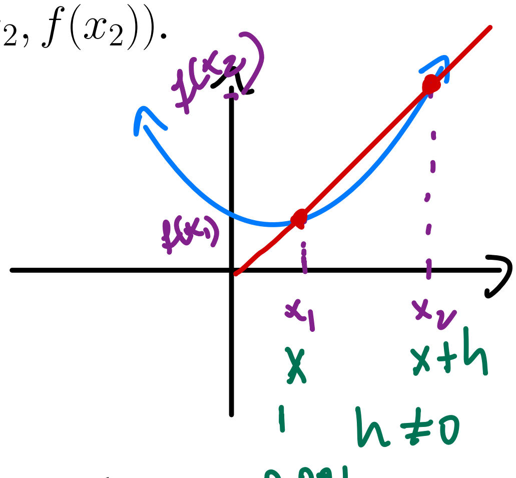
2. $f(-3) = 2(-3)^2 + 3(-3) + 1 = 2 \cdot 9 - 9 + 1 = 10$

3. $f(a) = 2a^2 + 3a + 1$

4. Difference Quotient

The difference quotient represents the average rate of change of a function between two values x and $x+h$.

Recall that the Average rate of change of a function f between two points x_1 and x_2 is just the slope of the line going through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



Average rate of change = slope of secant line

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Formula:

$$\boxed{\text{Difference quotient} = \frac{f(x+h) - f(x)}{(\cancel{x}+h) - \cancel{x})}}$$

$$\boxed{\text{Difference quotient} = \frac{f(x+h) - f(x)}{h}}$$ (*)!

Example 6. Find the difference quotient for $f(x) = 2x^2 + 3x + 1$.

$(x+h)$

$$\frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 + 3(x+h) + 1 - (2x^2+3x+1)}{h}$$

$$= \frac{2(x^2+2hx+h^2) + 3x+3h+1-2x^2-3x-1}{h}$$

$(x+h)^2 = (x+h)(x+h)$

$= x^2 + xh + hx + h^2$

$(x+h)^2 \neq x^2 + h^2$

$$= \frac{\cancel{2x^2}+4hx+2h^2+\cancel{3x}+3h\cancel{+1}\cancel{-2x^2}\cancel{-3x}\cancel{-1}}{h}$$

$$= \frac{4hx+2h^2+3h}{h} = \frac{\cancel{h}(4x+2h+3)}{\cancel{h}}$$

$$= 4x+3+2h \to 0$$

Goal is to cancel the h ~~out the~~ denominator

Example 7. Find the difference quotient for $f(x) = \dfrac{1}{x+1}$.

$$\frac{f(x+h) - f(x)}{h} \quad \frac{\frac{(x+1)}{(x+1)}\frac{1}{(x+h+1)} - \frac{1}{(x+1)}\frac{(x+h+1)}{(x+h+1)}}{h}$$

$$= \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h} = \frac{\frac{\cancel{x}\cancel{+1}-\cancel{x}-h\cancel{-1}}{(x+1)(x+h+1)}}{\frac{h}{1}}$$

$$= \frac{\frac{-h}{(x+1)(x+h+1)}}{\left(\frac{h}{1}\right)} = \frac{-\cancel{h}}{(x+1)(x+h+1)} \cdot \frac{1}{\cancel{h}}$$

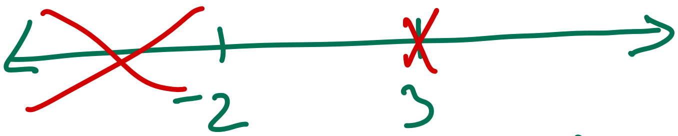
$$= \boxed{-\frac{1}{(x+1)(x+h+1)}}$$

Practice 1: Find the domain of functions:

1. $f(x) = \dfrac{\sqrt{2+x}}{3-x}$

→ $2+x$ can't be negative
→ denominator can't be zero
$3-x \neq 0$ so $x \neq 3$

Try these on your own!



Domain : $[-2,3) \cup (3,\infty)$

2. $g(x) = \sqrt[3]{x-1}$

Quantities unde (odd) roots can be negative. So domain is all real numbers

Domain $(-\infty, \infty)$

3. $h(x) = \dfrac{x^4}{(x-3)(x+3)}$ → denominator can't be zero.

$x \neq 3$, $x \neq -3$

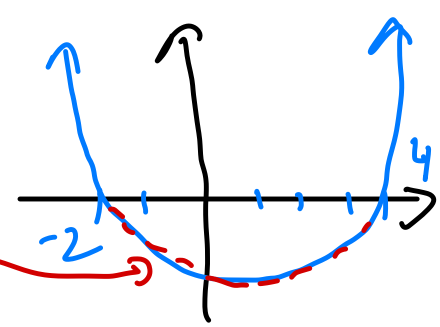
Domain $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

4. $f(t) = \sqrt{t^2 - 2t - 8}$

First factor the quantatity under square root.

$t^2 - 2t - 8 = (t+2)(t-4)$

$(t+2)(t-4)$ can't be negative



Domain : $(-\infty,-2] \cup [4,\infty)$

Practice 2: Evaluate (if possible) the function $g(x) = \dfrac{1-x}{1+x}$ at:

1. $g(2) = \dfrac{1-2}{1+2} = -1/3$

2. $g(-1) = \dfrac{1-(-1)}{1+(-1)}$ 0

not possible

3. $g(a-1) = \dfrac{1-(a-1)}{1+a-1} = \dfrac{2-a}{a}$

4. $g(x^2-1) = \dfrac{1-(x^2-1)}{1+x^2-1} = \dfrac{2-x^2}{x^2}$

Practice 3: Find the difference quotient for $f(x) = x^2 + 2x - 3$.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + 2(x+h) - 3 - (x^2+2x-3)}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h \cancel{-3} \cancel{-x^2} \cancel{-2x} \cancel{+3}}{h}$$

$$= \frac{2xh + h^2 + 2h}{h} = \frac{\cancel{h}(2x+h+2)}{\cancel{h}} = \boxed{2x+2+h}$$

Practice 4: Evaluate the piecewise defined function at the indicated values:

$$f(x) = \begin{cases} x^2 + 2x & x \leq -1 \\ x & -1 < x \leq 1 \\ -1 & 1 < x \end{cases}$$

1. $f(-4) = (-4)^2 + 2(-4) = 8$

2. $f(25) = -1$

3. $f(-1) = (-1)^2 + 2(-1) = 1 - 2 = -1$

4. $f(0) = 0$