1. Parabolas

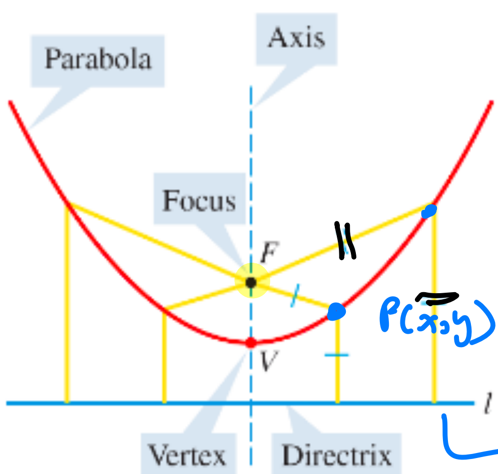
In this section we study parabolas from a geometric rather than an algebraic point of view. $x^2 = 4py$
We begin with the geometric definition of a parabola and show how this leads to the algebraic formula that we are already familiar with.

$y = ax^2$

GEOMETRIC DEFINITION OF A PARABOLA

same distance

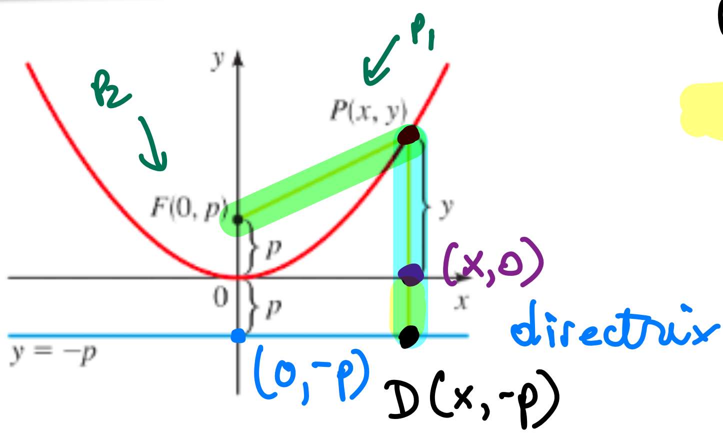
A **parabola** is the set of all points in the plane that are equidistant from a fixed point *F* (called the **focus**) and a fixed line *l* (called the **directrix**).



horizontal line l

2. Deriving the equation of the parabola

We will concentrate on parabolas that are situated with the vertex at the origin and that have a vertical or horizontal axis of symmetry.



$P_1(x_1, y_1)$ & $P_2(x_2, y_2)$

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(P, F) = d(P, \text{directrix})$$

$$d(P, F) = \sqrt{(0 - x)^2 + (p - y)^2} = \sqrt{x^2 + (p - y)^2}$$

$$d(P, \text{directrix}) = d(P, D) = \sqrt{(x - x)^2 + (-p - y)^2}$$

P_1 (x_1, y_1) P_2 (x_2, y_2)

$$= \sqrt{(p + y)^2}$$

$$\left(\sqrt{x^2 + (p - y)^2}\right)^2 = \left(\sqrt{(p + y)^2}\right)^2 \quad / \text{ square both sides}$$

$$x^2 + (p - y)^2 = (p + y)^2$$

$$x^2 + (p - y)(p - y) = (p + y)(p + y)$$

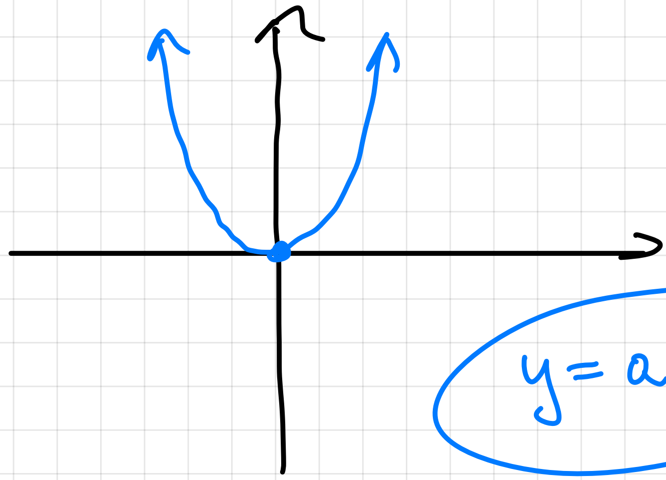
$$x^2 + p^2 - py - yp + y^2 = p^2 + py + yp + y^2$$

$$x^2 + p^2 - 2py + y^2 = p^2 + 2py + y^2$$

$+2py \qquad +2py$

$$x^2 + \cancel{p^2} + \cancel{y^2} = \cancel{p^2} + 4py + \cancel{y^2}$$

$$\boxed{x^2 = 4py}$$ → equation of parabola with vertex at origin and focus on y axis.



$y = x^2$

$y = ax^2 + bx + c$

$y = ax^2$

$\frac{x^2}{4p} = \frac{4py}{4p}$ → solve for y

$\frac{1}{4p}x^2 = y$ or $y = \frac{1}{4p}x^2$

PARABOLA WITH VERTICAL AXIS

The graph of the equation

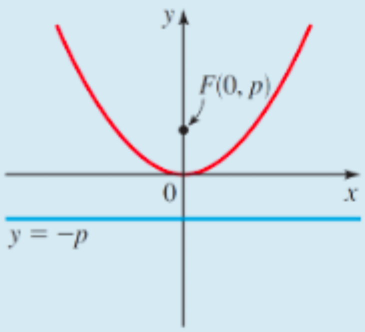
$$x^2 = 4py$$

is a parabola with the following properties.

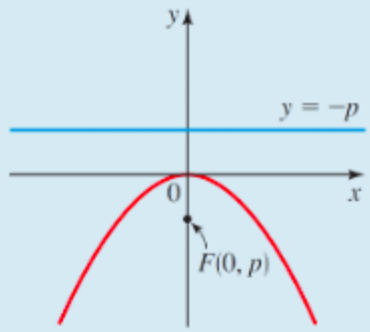
VERTEX	$V(0, 0)$
FOCUS	$F(0, p)$
DIRECTRIX	$y = -p$

The parabola opens upward if $p > 0$ or downward if $p < 0$.

Focal diameter $|4p|$



$x^2 = 4py$ with $p > 0$



$x^2 = 4py$ with $p < 0$

PARABOLA WITH HORIZONTAL AXIS

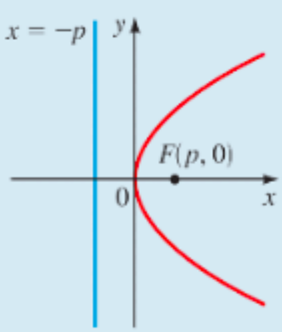
The graph of the equation

$$y^2 = 4px$$

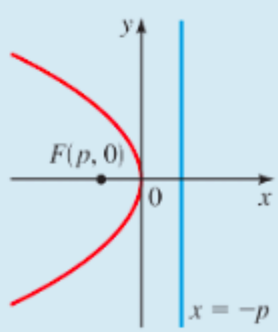
is a parabola with the following properties.

VERTEX	$V(0, 0)$
FOCUS	$F(p, 0)$
DIRECTRIX	$\boxed{x = -p}$ vertical line

The parabola opens to the right if $p > 0$ or to the left if $p < 0$.



$y^2 = 4px$ with $p > 0$



$y^2 = 4px$ with $p < 0$

$F(0,p)$

Example 1: Find an equation for the parabola with vertex $V(0,0)$ and focus $F(0,2)$, and sketch its graph.

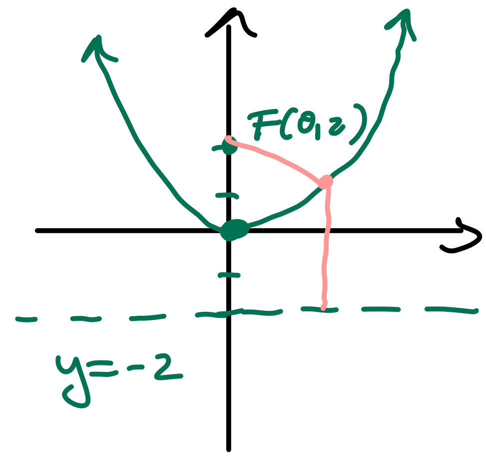
$\downarrow$ parabola opening up

$x^2 = 4py$

$x^2 = 4(2)y$

$\boxed{x^2 = 8y}$

$\frac{x^2}{8} = y \Rightarrow \boxed{y = \frac{1}{8}x^2}$



Example 2: Find the focus, focal diameter and directrix of the parabola $y = -x^2$, and sketch the graph.

(1) $x^2 = 4py$

or

(2) $y^2 = 4px$

$\downarrow$

$-x^2 = y$

$\boxed{x^2 = -y}$

$x^2 = -1y$

$x^2 = 4py$

$4p = -1$

$\boxed{p = -1/4}$

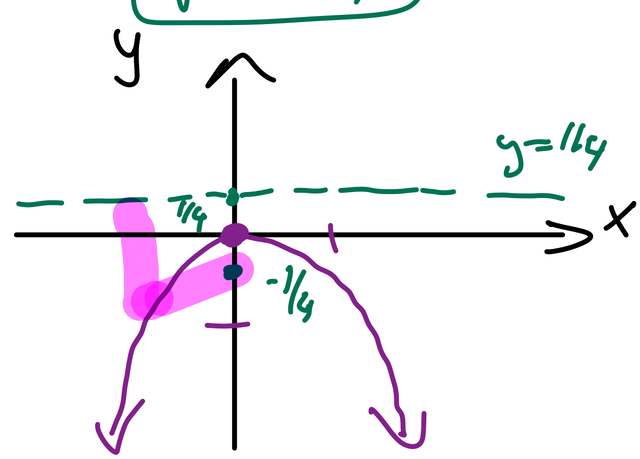
Focus: $(0,p) = (0,-1/4)$

Focal diameter: $|4p| = |4(-\frac{1}{4})| = 1$

Directrix: $y = -p$

$y = -(-1/4)$

$\boxed{y = 1/4}$



Example 3: A parabola has the equation $\boxed{6x + y^2 = 0.}$

(a) Find the focus, focal diameter and directrix of the parabola.

$6x + y^2 = 0$ bring it to form 2

① $x^2 = 4py$ or ② $y^2 = 4px$

$-6x \quad -6x$

$\boxed{y^2 = -6x}$

$\frac{-6}{4} = \frac{4p}{4}$

$\boxed{p = -3/2}$ → we know parabola opens to the left since $p = -3/2 < 0$.

Focus: $F(p, 0) = (-3/2, 0)$

Focal diameter: $|4p| = |4(-3/2)| = |-6| = 6$

Directrix: $x = -p$

$x = -(-3/2)$

$x = 3/2$

b)