Name: ______________________________ Total: 115 points

Show all work. Remember the constant of integration and use absolute values where appropriate.

1. (12 points) For each integral, state the technique you would use to **start** the problem. Do **not** evaluate the integral.

(a) $\int w \ln(w)\, dw$

Answer: Integration by parts with $u = \ln w$, $dv = w\, dw$.

(b) $\int 8 \sin^2(x) \cos^3(x)\, dx$

Answer: Trig integral: odd power of cosine; save one $\cos x$, write $\cos^2 x = 1 - \sin^2 x$, let $u = \sin x$.

(c) $\int \frac{x^2}{\sqrt{49 - x^2}}\, dx$

Answer: Trig substitution $x = 7 \sin \theta$.

(d) $\int \frac{3}{(x-1)(x+2)}\, dx$

Answer: Partial fraction decomposition.

(e) $\int \frac{x}{\sqrt{x^2 - 5}}\, dx$

Answer: Ordinary u-substitution, $u = x^2 - 5$ (no trig sub needed).

(f) $\int \frac{4e^{2x}}{e^{2x} + 14e^x + 45}\, dx$

Answer: Substitution $u = e^x$ (so $dx = du/u$) to get a rational function, then partial fractions.

2. (9 points) Write out the **form** of the partial fraction decomposition of each integrand. Do **not** solve for the coefficients and do **not** evaluate the integral.

   (a) (3 points) $\displaystyle\int \frac{x^5+64}{(x^2-x)(x^4+16x^2+64)}\,dx$

   Answer: $\displaystyle\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+8}+\frac{Ex+F}{(x^2+8)^2}$

   (b) (3 points) $\displaystyle\int \frac{x^2}{x^2+x-20}\,dx$

   Answer: Long division first: $\displaystyle 1+\frac{A}{x-4}+\frac{B}{x+5}$

   (c) (3 points) $\displaystyle\int \frac{26}{(x-1)(x^2+25)}\,dx$

   Answer: $\displaystyle\frac{A}{x-1}+\frac{Bx+C}{x^2+25}$

3. (8 points) Evaluate the integral.

$$\int (x^2 + 7x)\cos(x)\,dx$$

Answer: Integrate by parts with $u = x^2+7x$, $dv = \cos x\,dx$, so $du = (2x+7)\,dx$, $v = \sin x$:

$$\int (x^2 + 7x)\cos x\,dx = (x^2 + 7x)\sin x - \int (2x+7)\sin x\,dx.$$

Integrate by parts again with $u = 2x + 7$, $dv = \sin x\,dx$, so $du = 2\,dx$, $v = -\cos x$:

$$\int (2x+7)\sin x\,dx = -(2x+7)\cos x + \int 2\cos x\,dx = -(2x+7)\cos x + 2\sin x.$$

Therefore

$$\int (x^2 + 7x)\cos x\,dx = (x^2 + 7x)\sin x + (2x+7)\cos x - 2\sin x + C.$$

4. (10 points) Evaluate the integral.

$$\int e^{8\theta}\sin(9\theta)\,d\theta$$

Answer: Let $I = \int e^{8\theta}\sin(9\theta)\,d\theta$. Integrate by parts with $u = \sin(9\theta)$, $dv = e^{8\theta}\,d\theta$, so $du = 9\cos(9\theta)\,d\theta$, $v = \frac{1}{8}e^{8\theta}$:

$$I = \tfrac{1}{8}e^{8\theta}\sin(9\theta) - \tfrac{9}{8}\int e^{8\theta}\cos(9\theta)\,d\theta.$$

Integrate by parts again with $u = \cos(9\theta)$, $dv = e^{8\theta}\,d\theta$, so $du = -9\sin(9\theta)\,d\theta$, $v = \frac{1}{8}e^{8\theta}$:

$$\int e^{8\theta}\cos(9\theta)\,d\theta = \tfrac{1}{8}e^{8\theta}\cos(9\theta) + \tfrac{9}{8}I.$$

Substituting back,

$$I = \tfrac{1}{8}e^{8\theta}\sin(9\theta) - \tfrac{9}{64}e^{8\theta}\cos(9\theta) - \tfrac{81}{64}I \quad\Longrightarrow\quad \tfrac{145}{64}I = \tfrac{1}{64}e^{8\theta}\big(8\sin(9\theta) - 9\cos(9\theta)\big).$$

Therefore

$$\int e^{8\theta}\sin(9\theta)\,d\theta = \frac{1}{145}e^{8\theta}\big(8\sin(9\theta) - 9\cos(9\theta)\big) + C.$$

5. (7 points) Evaluate the integral.

$$\int_0^{\pi/2} \sin^3(y)\cos^4(y)\,dy$$

Answer: The power of sine is odd, so save one $\sin y$ and use $\sin^2 y = 1-\cos^2 y$:

$$\int_0^{\pi/2} \sin^3 y\cos^4 y\,dy = \int_0^{\pi/2} (1-\cos^2 y)\cos^4 y\,\sin y\,dy.$$

Let $u=\cos y$, $du=-\sin y\,dy$. When $y=0$, $u=1$; when $y=\pi/2$, $u=0$:

$$= -\int_1^0 (u^4-u^6)\,du = \int_0^1 (u^4-u^6)\,du = \left[\frac{u^5}{5}-\frac{u^7}{7}\right]_0^1 = \frac{1}{5}-\frac{1}{7} = \frac{2}{35}.$$

6. (7 points) Evaluate the integral.

$$\int \tan^3(x)\sec^6(x)\,dx$$

Answer: The power of secant is even, so save $\sec^2 x$ and write $\sec^4 x = (1+\tan^2 x)^2$:

$$\int \tan^3 x\sec^6 x\,dx = \int \tan^3 x\,(1+\tan^2 x)^2\sec^2 x\,dx.$$

Let $u=\tan x$, $du=\sec^2 x\,dx$:

$$= \int u^3(1+2u^2+u^4)\,du = \int (u^3+2u^5+u^7)\,du = \frac{u^4}{4}+\frac{u^6}{3}+\frac{u^8}{8}+C.$$

Therefore

$$\int \tan^3 x\sec^6 x\,dx = \frac{1}{8}\tan^8 x+\frac{1}{3}\tan^6 x+\frac{1}{4}\tan^4 x+C.$$

7. (6 points) For each integral, state the trigonometric substitution you would use to get started. Do **not** evaluate the integrals.

   (a) (2 points) $\displaystyle\int \frac{x^3}{\sqrt{16+x^2}}\,dx$

   Answer: $x = 4\tan\theta$

   (b) (2 points) $\displaystyle\int \frac{\sqrt{4x^2-25}}{x}\,dx$

   Answer: $x = \frac{5}{2}\sec\theta$

   (c) (2 points) $\displaystyle\int \frac{\sqrt{x^2-9}}{x^4}\,dx$

   Answer: $x = 3\sec\theta$

8. (10 points) Choose **one** of the three integrals from Question 7 and evaluate it completely. Include a labeled right triangle and write your final answer in terms of x.

   I am solving part: ____________

Answer: (a) Let $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$, $\sqrt{16+x^2} = 4\sec\theta$. Triangle: opposite x, adjacent 4, hypotenuse $\sqrt{x^2+16}$, so $\sec\theta = \frac{\sqrt{x^2+16}}{4}$.

$$\begin{aligned}
\int \frac{x^3}{\sqrt{16+x^2}}\,dx &= \int \frac{64\tan^3\theta \cdot 4\sec^2\theta}{4\sec\theta}\,d\theta = 64\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta \\
&= 64\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C \qquad (w = \sec\theta) \\
&= \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16} + C.
\end{aligned}$$

(b) Let $x = \frac{5}{2}\sec\theta$, $dx = \frac{5}{2}\sec\theta\tan\theta\,d\theta$, $\sqrt{4x^2-25} = \sqrt{25\sec^2\theta - 25} = 5\tan\theta$. Triangle:

hypotenuse $2x$, adjacent 5, opposite $\sqrt{4x^2-25}$, so $\tan\theta = \frac{\sqrt{4x^2-25}}{5}$ and $\theta = \sec^{-1}\left(\frac{2x}{5}\right)$.

$$\int \frac{\sqrt{4x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{\frac{5}{2}\sec\theta} \cdot \tfrac{5}{2}\sec\theta\tan\theta\,d\theta = 5\int \tan^2\theta\,d\theta = 5\int (\sec^2\theta - 1)\,d\theta$$
$$= 5\tan\theta - 5\theta + C = \sqrt{4x^2-25} - 5\sec^{-1}\left(\frac{2x}{5}\right) + C.$$

(c) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$. Triangle: hypotenuse x, adjacent 3, opposite $\sqrt{x^2-9}$, so $\sin\theta = \frac{\sqrt{x^2-9}}{x}$.

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{3\tan\theta \cdot 3\sec\theta\tan\theta}{81\sec^4\theta}\,d\theta = \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{9}\int \sin^2\theta\cos\theta\,d\theta$$
$$= \frac{1}{27}\sin^3\theta + C \qquad (w = \sin\theta)$$
$$= \frac{(x^2-9)^{3/2}}{27x^3} + C.$$

9. (9 points) Evaluate the integral.

$$\int \frac{2}{x^3+x^2}\,dx$$

Answer: Factor the denominator: $x^3+x^2 = x^2(x+1)$. Decompose

$$\frac{2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \quad \Longrightarrow \quad 2 = Ax(x+1) + B(x+1) + Cx^2.$$

$x=0$: $B=2$. $x=-1$: $C=2$. Coefficient of x^2: $A+C=0$, so $A=-2$. Then

$$\int \frac{2}{x^3+x^2}\,dx = \int \left(-\frac{2}{x} + \frac{2}{x^2} + \frac{2}{x+1}\right) dx = -2\ln|x| - \frac{2}{x} + 2\ln|x+1| + C.$$

10. (8 points) Make a substitution to express the integrand as a rational function, and then evaluate the integral.

$$\int_2^4 \frac{dx}{x\sqrt{x-1}}$$

Answer: Let $u=\sqrt{x-1}$, so $x=u^2+1$ and $dx = 2u\,du$. When $x=2$, $u=1$; when $x=4$, $u=\sqrt{3}$:

$$\int_2^4 \frac{dx}{x\sqrt{x-1}} = \int_1^{\sqrt{3}} \frac{2u}{(u^2+1)\,u}\,du = 2\int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2\Big[\tan^{-1} u\Big]_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}.$$

11. (4 points) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^\infty 4x^{-5}\,dx$$

Answer:

$$\int_1^\infty 4x^{-5}\,dx = \lim_{t\to\infty}\int_1^t 4x^{-5}\,dx = \lim_{t\to\infty}\left[-x^{-4}\right]_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t^4}\right) = 1.$$

The integral is **convergent** and equals 1.

12. (10 points) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it. Use proper limit notation.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx$$

Answer: First find the antiderivative by parts with $u = \ln x$, $dv = x^{-2}\,dx$, so $du = \frac{1}{x}\,dx$, $v = -\frac{1}{x}$:

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Then

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1 - \frac{\ln t}{t} - \frac{1}{t}\right).$$

By L'Hôpital's rule, $\lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{1/t}{1} = 0$, and $\frac{1}{t}\to 0$. The integral is **convergent** and equals 1.