Day 15

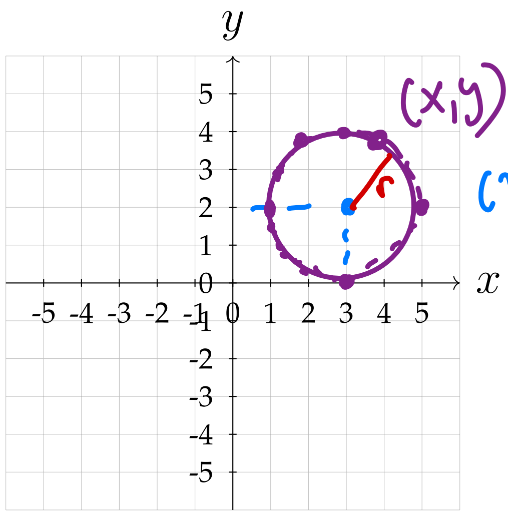
9.1 The Circles and 9.2 The Ellipse

1. Circles

Next, we will start graphing specific categories of equations such as circles, ellipses, hyperbolas and parabola. We begin with circles.

same distance

Definition: A circle is the set of all points in a plane that are equidistant from a fixed point called ____center____. The fixed distance from any point on the circle to the center is called the ____radius____.



r - radius

r = 2

center : (h, k)

k → y coord of center

h → x coordinate of center

Standard form of an equation of a circle

Given a circle centered at (h, k) with radius r, the standard form is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

↓ standard formula for a circle

Example 1

center: $(4, -3)$
radius: $r = 5$

1. $(x-4)^2 + (y+3)^2 = 25$

$(x-4)^2 + (y-(-3))^2 = 5^2$

center: $(0, 1/2)$
radius: $r = \sqrt{12} = \sqrt{4 \cdot 3}$
$r = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

2. $x^2 + (y - 1/2)^2 = 12$

$(x-0)^2 + (y - 1/2)^2 = (\sqrt{12})^2$

3. $x^2 + y^2 = 7$

$(x-0)^2 + (y-0)^2 = (\sqrt{7})^2$

$(0,0)$
$r = \sqrt{7}$

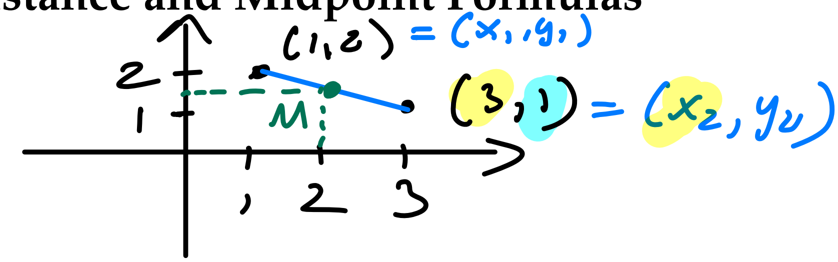
Example 2 Write the standard form of an equation of a circle with center $(-4, 6)$ and radius 2.

$(x-h)^2 + (y-k)^2 = r^2$

$(x-(-4))^2 + (y-6)^2 = 2^2$

$$\boxed{(x+4)^2 + (y-6)^2 = 4}$$

2. Recall: Distance and Midpoint Formulas



Distance Formula: The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$d = \sqrt{(3-1)^2 + (1-2)^2} = \sqrt{4+1} = \sqrt{5}$

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

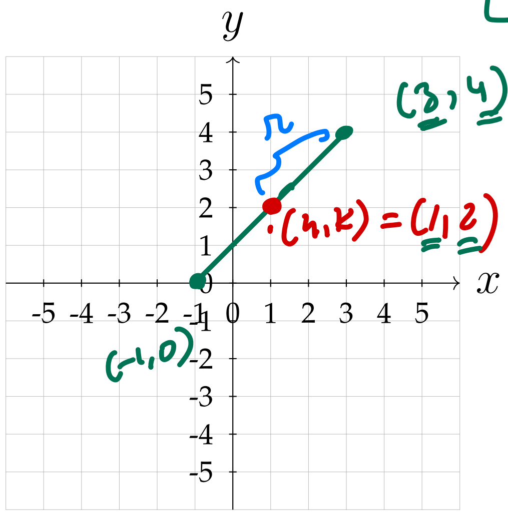
Midpoint Formula: The midpoint $M(x, y)$ of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is:

$$\boxed{M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$$

$M\left(\frac{1+3}{2}, \frac{2+1}{2}\right) = \left(\frac{4}{2}, \frac{3}{2}\right) = (2, 1.5)$

Example 3 Write the standard form of an equation of a circle with the endpoints of the diameter as $(-1, 0)$ and $(3, 4)$. Graph the circle.

→ twice the radius



$(x-h)^2 + (y-k)^2 = r^2$

$(h,k) = M\left(\frac{-1+3}{2}, \frac{0+4}{2}\right) = (1,2)$

$(x-1)^2 + (y-2)^2 = (\sqrt{8})^2$

$r = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8} = \sqrt{4 \cdot 2}$

$r = 2\sqrt{2}$

$(x-1)^2 + (y-2)^2 = 8$

General form of an Equation of a circle

$(h,k) = ?$
$r = ?$

An equation of a circle written in the form $x^2 + y^2 + Ax + By + C = 0$ is called the general form of an equation of a circle.

Example 4 Write the equation of the circle in standard form by **completing the square.**

$\boxed{x^2 + y^2 + 10x - 6y + 25 = 0}$ general form → $\boxed{(x-h)^2 + (y-k)^2 = r^2}$ standard form

1) $\underbrace{x^2 + 10x}_{\text{binomial in } x} + \underbrace{y^2 - 6y}_{\text{binomial in } y} = -25$

2) $x^2 + 10x + 25 + y^2 - 6y + 9 = -25 + 25 + 9$

take half of coeff of x, square it and add to both sides

$\left(\frac{10}{2}\right)^2 = 5^2 = 25$

$\left(-\frac{6}{2}\right)^2 = (-3)^2 = 9$

3) $(x+5)^2 + (y-3)^2 = 9$

Steps for completing the square

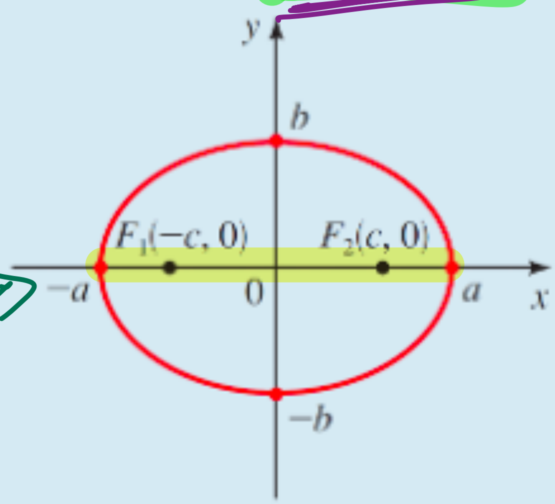
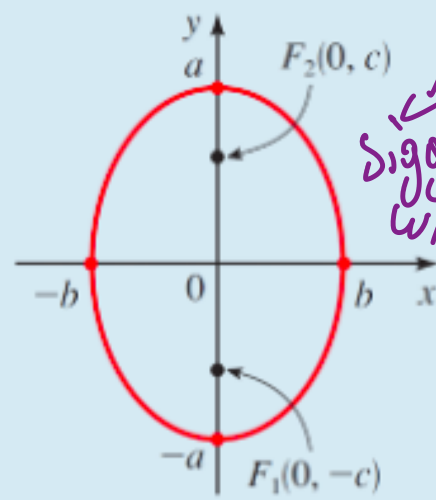
1. Rewrite the general equation by grouping the x terms together and y terms together
2. Complete the square for binomial in x and for binomial in y.

3. Equations and Graphs of ellipses

The following box summarizes the equation and features of an ellipse centered at the origin.

ELLIPSE WITH CENTER AT THE ORIGIN

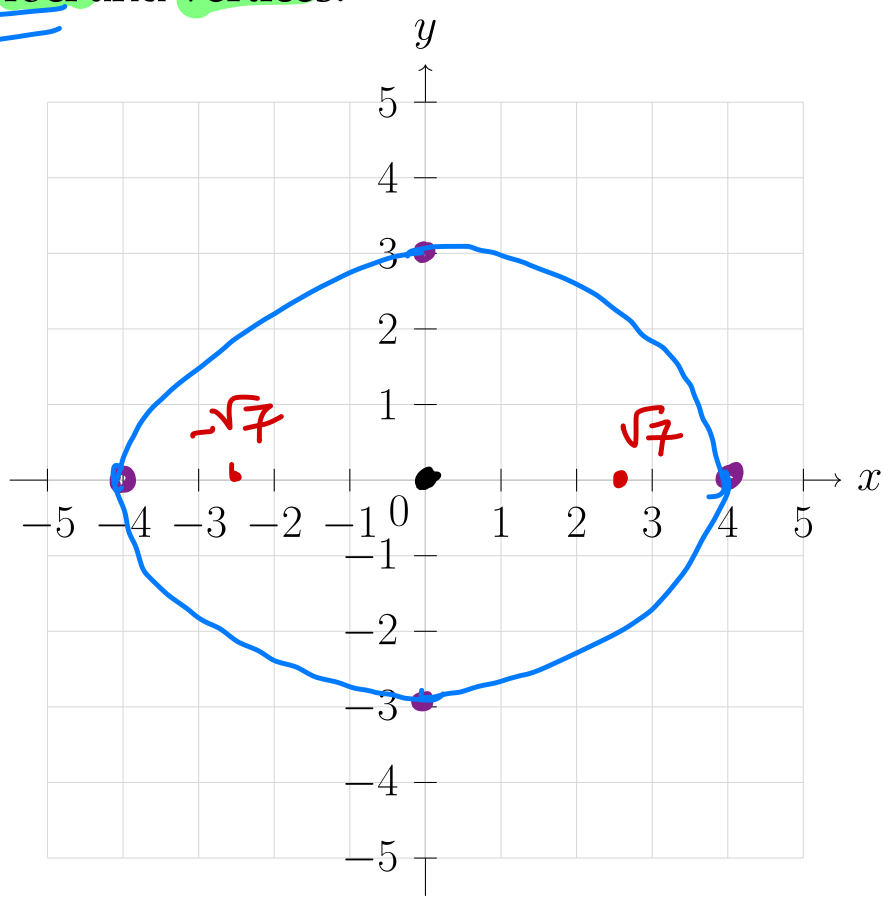
The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ $a > b > 0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2 = a^2 - b^2$	$(0, \pm c)$, $c^2 = a^2 - b^2$
GRAPH		

bigger number with the x^2 term

bigger # w/ y^2

Example 5: Given $\boxed{\frac{x^2}{16} + \frac{y^2}{9} = 1}$, graph the ellipse and identify the center, foci and vertices.



- elongated x axis

- foci :

- vertex : $(4,0)$ 4
$(-4,0)$

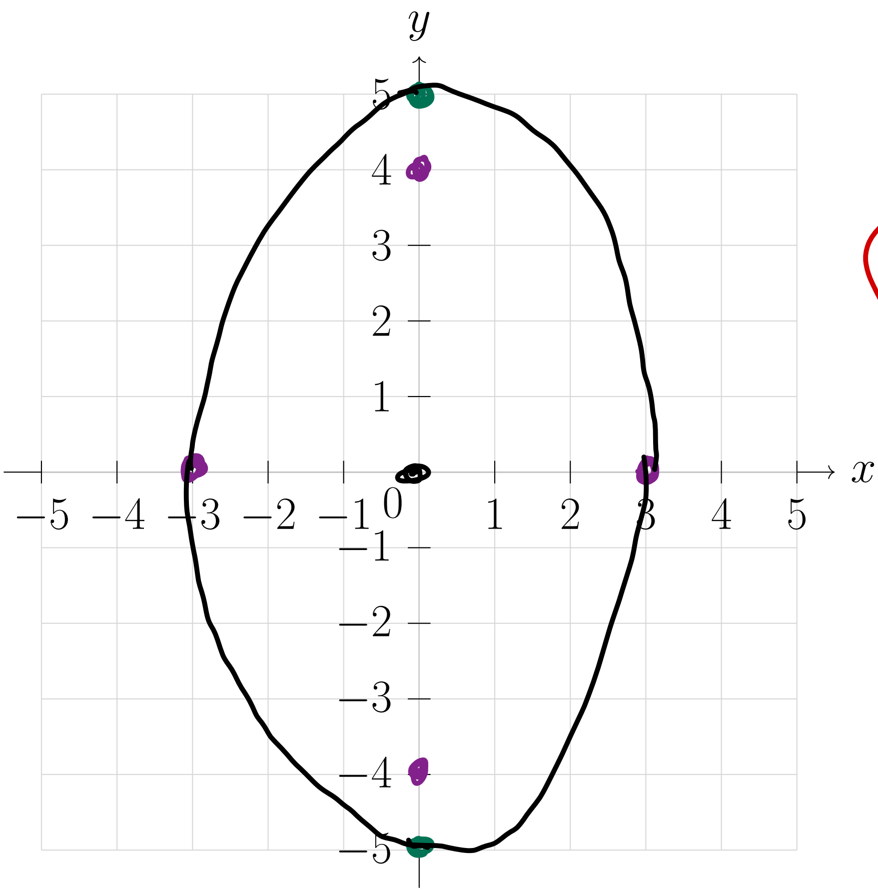
- foci : $(\sqrt{7},0)$, $(-\sqrt{7},0)$

$c^2 = a^2 - b^2$
$= 16 - 9 = 7$
$c = \sqrt{7}$

$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ compare $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a = 4 \rightarrow$ vertex
$b = 3 \rightarrow$ covertex

Example 6: Find the foci of the ellipse $25x^2 + 9y^2 = 225$, and sketch its graph.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

divide by 225

$$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

$$\boxed{\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1}$$

elongated along y axis

- vertices : $(0, a)$ & $(0, -a)$: $(0,5), (0,-5)$
- foci :

$$c^2 = a^2 - b^2$$

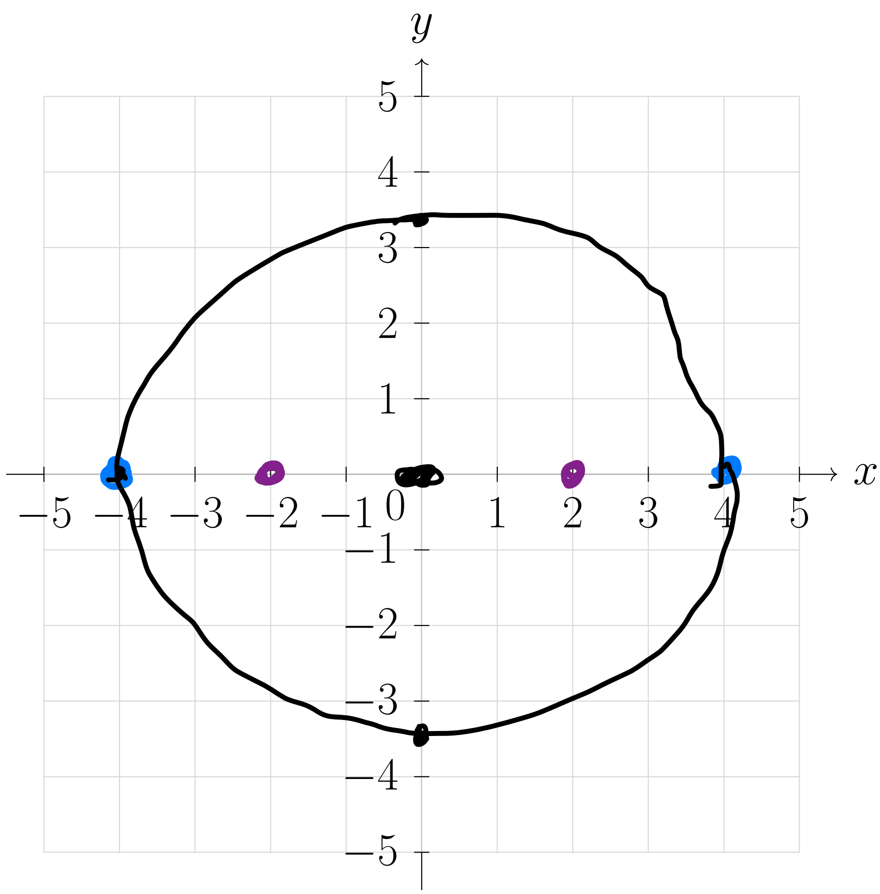
$$c^2 = 5^2 - 3^2 = 25 - 9 = 16$$

$$c = 4$$

- covertices : $(b,0)$ & $(-b,0)$

$(3,0)$ & $(-3,0)$

Example 7: The vertices of an ellipse are $(\pm 4, 0)$, and the foci are $(\pm 2, 0)$. Find its equation, and sketch the graph.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a = 4$

$c = 2$

$c^2 = a^2 - b^2$

$4 = 16 - b^2$

$-12 = -b^2$

$b^2 = 12$

$b = \sqrt{12} \approx 3.46$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{12})^2} = 1$$

$$\boxed{\frac{x^2}{16} + \frac{y^2}{12} = 1} \rightarrow \frac{y^2}{12} = 1 - \frac{x^2}{16}$$

$$y^2 = 12\left(1 - \frac{x^2}{16}\right)$$

Calculator instructions

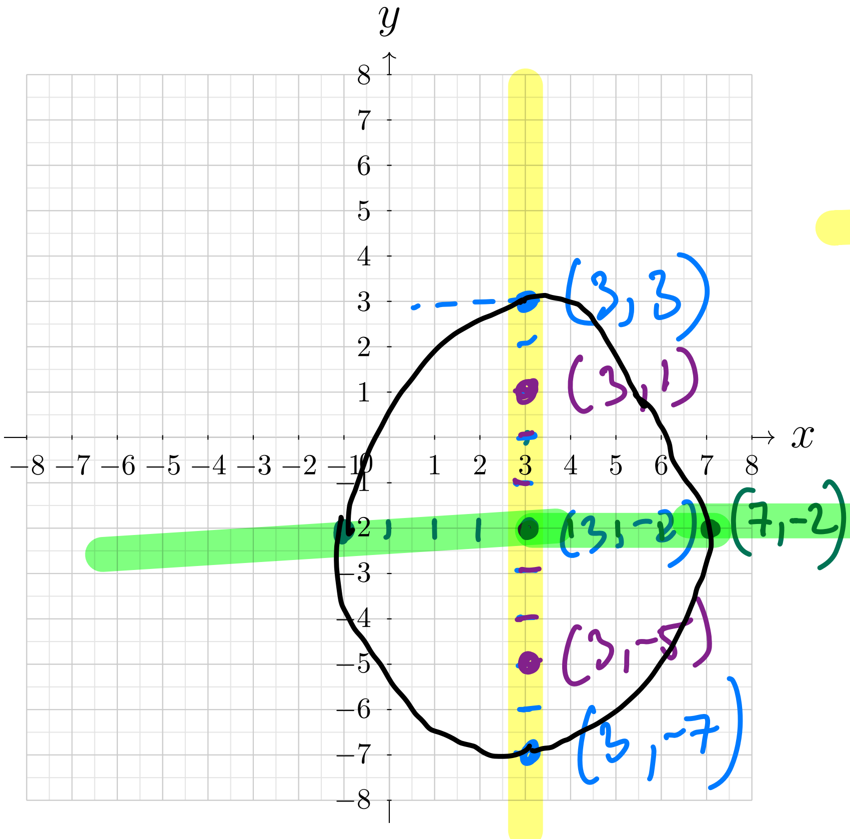
1) solve for y

2) $y = \pm\sqrt{12\left(1 - \frac{x^2}{16}\right)}$

4. Graphing and Ellipse centered at (h, k)

Replacing x by $x - h$ and y by $y - k$ shifts the graph of an ellipse h units horizontally and k units vertically.

Example 8: Given $\dfrac{(x-3)^2}{16} + \dfrac{(y+2)^2}{25} = 1$, graph the ellipse and identify the center, foci and vertices.



center: $(3,-2)$

elongated vertically

$\boxed{a=5}$

$b=4$

vertex: $(3,3)$ & $(3,-7)$

foci: $(3,1)$ & $(3,-7)$

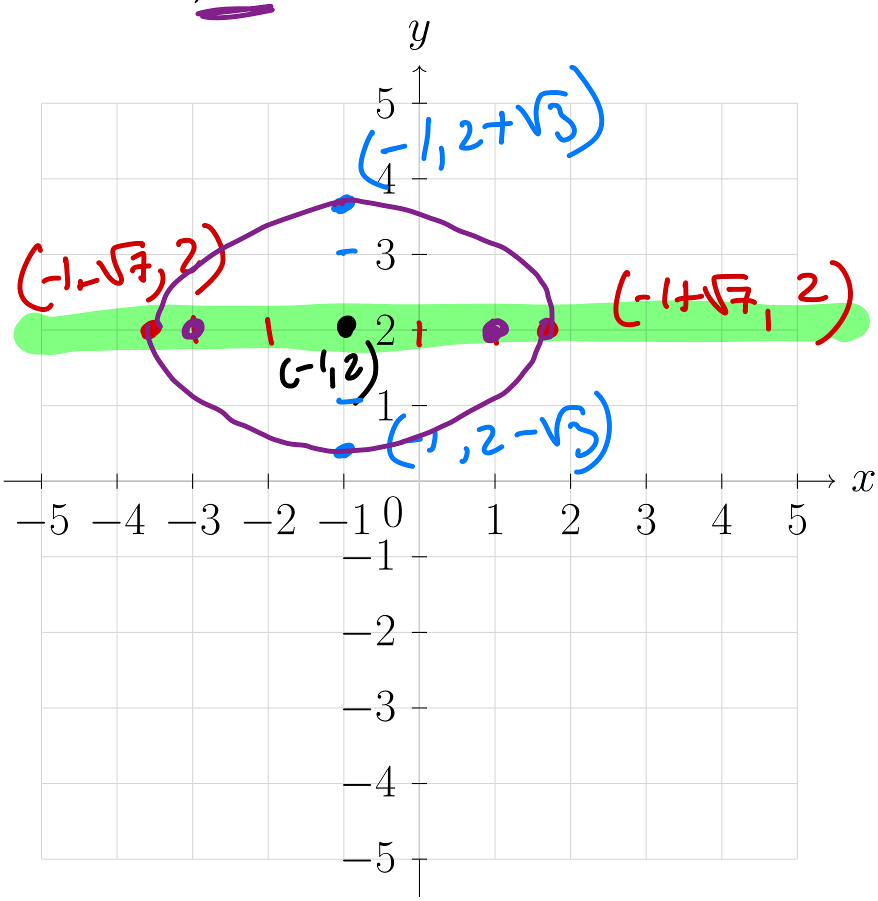
need c

$c^2 = a^2 - b^2$

$= 25 - 16 = 9$

$c = 3$

Example 9: Practice: Given $3x^2 + 7y^2 + 6x - 28y + 10 = 0$, write the equation of the ellipse in standard form, graph the ellipse and identify the center, foci and vertices.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- elongated horizontally
- center $(-1, 2)$
- foci $c^2 = a^2 - b^2$
  $= 7 - 3$
  $= 4$

$$3x^2 + 7y^2 + 6x - 28y + 10 = 0$$

$$3x^2 + 6x + 7y^2 - 28y = -10$$

$$3(x^2 + 2x + 1) + 7(y^2 - 4y + 4) = -10 + 3 + 28$$

$$\frac{3(x+1)^2 + 7(y-2)^2}{21} = \frac{21}{21}$$

$$\boxed{\frac{(x+1)^2}{7} + \frac{(y-2)^2}{3} = 1}$$

$7 \rightarrow a = \sqrt{7}$

$3 \rightarrow b = \sqrt{3}$