Thursday, April 17th

Finished Polynomials

Started Rational Functions

Next Written HW

Group Practice:

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.

   (a) Does $x - 2$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(2)$.

$$
\begin{array}{r|l}
 & x^2 - 7x + 12 \\
\hline
x-2 & x^3 - 9x^2 + 26x - 24 \\
 & x^3 - 2x^2 \\
\hline
 & -7x^2 + 26x - 24 \\
 & -7x^2 + 14x \\
\hline
 & 12x - 24 \\
 & 12x - 24 \\
\hline
 & 0
\end{array}
$$

$f(2) = 2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 0$

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$

   (a) Does $x - 3$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(3)$.

$x - 3 \overline{) x^4 - 5x^3 + 2x^2 + 8x - 6}$

$r = -18$

$f(3) = -18$

The Remainder and Factor Theorem

Remainder Theorem:
If a polynomial $f(x)$ is divided by $x - a$, then the remainder is equal to $f(a)$.

$f(x) \div (x-2)$ has a remainder of $f(2)$.

Factor Theorem:
A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

$x^3 - 9x^2 + 26x - 24 = (x-2)(x^2 - 7x + 12)$
$= (x-2)(x-3)(x-4)$

3.4 The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

factors of 12: $\pm 1, \pm 12$
$\pm 2, \pm 6$
$\pm 3, \pm 4$

$12 = (-1)(-12)$

factors of 25: ± 5
$\pm$, ± 25

The Rational Zero Test

leading coefficient

constant coefficient

If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and
q is a factor of the leading coefficient a_n.

Possible rational zeros = $\dfrac{\text{factors of constant term}}{\text{factors of leading term}}$

Example 4: Find the rational zeros of

Graph it w/ calculator

1. $f(x) = 1x^3 - 3x^2 + 4.$

coeff : 1, −3, 4

$f(-1) = (-1)^3 - 3(-1)^2 + 4 = -1-3+4 = 0$

$f(1) = 1-3+4 \neq 0$

$f(2) = 2^3 - 3\cdot 2^2 + 4 = 8-12+4 = 0$

leading coeff = 1

const coeff = 4

$\frac{\pm 1}{\pm 1} = \pm 1$

factors constant coeff 4 : ±1, ±2, ±4

$\frac{\pm 2}{\pm 1} = \pm 2$

factors leading coeff 1 : ±1

$\frac{\pm 4}{\pm 1} = \pm 4$

possible rational zeros : ±1, ±2, ±4

$f(-1) = 0$, ~~$f(1) \neq 0$~~, $f(2) = 0$ $f(-2) \neq 0$ $f(4) \neq 0$ $f(-4) \neq 0$

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

constant coeff : 3

factors of 3 : ±1, ±3

leading coeff : 2

factor of 2 : ±1, ±2

possible rational zeros : ±1, ±1/2, ±3, ±3/2

actual rational zeros : −3, 1/2, 1

factorazition
$h(x) = (x+3)(x-1)(2x-1)$

3. $g(x) = x^3 - 2$

a $\quad b = \sqrt[3]{2}$

$g(x) = x^3 - (\sqrt[3]{2})^3$

$g(x) = \underbrace{(x - \sqrt[3]{2})}(x^2 + x\sqrt[3]{2} + (\sqrt[3]{2})^2)$

$x - \sqrt[3]{2} = 0$

$x = \sqrt[3]{2} = 2^{1/3} = 1.259\ldots$ irrational

Difference of cubes:

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$2 = (\sqrt[3]{2})^3$

$2 = (\sqrt{2})^2$

2.5 The Fundamental Theorem of Algebra

A nth degree polynomial can have at most n real roots.
In the complex number system , every nth degree polynomial has exactly n zeros.

Types of Zeros of a polynomial function f

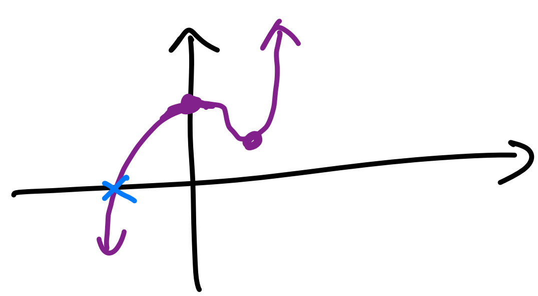
- Real
  - rational zeros : $1, 2, 7/2, -1/4, \ldots$
  - irrational zeros : $\pi, \sqrt{2}, \sqrt[3]{2}, e, \ldots$

zeros are actual x intercepts

- Complex : $a + bi$

ex: $1 - 2i$ its conjugate is $1 + 2i$

ex:



min degree is 3

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:

$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot ... \cdot (x - r_n)$

where $r_1, r_2, r_3, ..., r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.

$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$

$x^2 + 4 = ?$

$\cancel{(x-2)(x+2)}$

$\cancel{(x+2)(x+2)}$

$\cancel{(x-2)(x-2)}$

$x^2 + 4 = 0$

$x^2 = -4 \Rightarrow x = \pm\sqrt{-4}$

$x = \pm 2i$

$\boxed{f(x) = (x-1)(x-2i)(x+2i)(x+\sqrt{3})(x-\sqrt{3})}$

$x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$f(2i) = (2i-1)(2i-2i)(2i+2i)(2i+\sqrt{3})(2i-\sqrt{3}) = 0$

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.

$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$

Step 1 Find possible rational zeros

Step 2 Find actual rational zeros

Step 3 Use long divission to find zeros

Step 4 Complete factorization

1. Rational Functions

In this section we will review rational functions and concentrate on how to graph them by hand.

A rational function is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. Even though rational fractions are constructed from polynomials, their graphs look quite different from the graphs of polynomial functions.

Examples of rational functions:
For each function, decide if it is a rational function and then if it is find its domain, x-intercepts and y-intercept.

1. $f(x) = \dfrac{x^2+3}{(x-1)(x+2)}$.

yes

2. $f(x) = \dfrac{x^2-3}{(x^2+1)(x+2)}$. yes

3. $f(x) = \dfrac{x^2+3}{(\sqrt{x}-1)(x+2)}$. $\dfrac{\text{polynomial}}{\text{not polynomial}}$

2

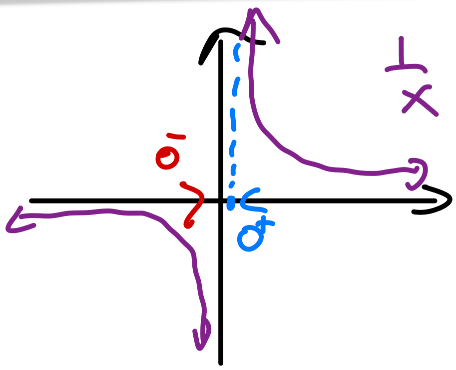
$\sqrt{x} = x^{1/2}$

The **domain of a rational function** consists of all real numbers x except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those x-values.

2. A simple rational Function $f(x) = \dfrac{1}{x}$

reciprocal

Graph the rational function $f(x) = \dfrac{1}{x}$ and state its domain and range.



Domain: $-\infty$ 0 ∞

$(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

horizontal asymptotes

End behaviour: $x \to \infty$, $f(x) \to 0$ $\quad$ $x \to -\infty$, $f(x) \to 0$

Behaviour near $x=0$

vertical asymptotes

$\begin{cases} x \to 0^+ \\ f(x) \to \infty \end{cases}$ approaches 0 from the right

$\begin{cases} x \to 0^- \\ f(x) \to -\infty \end{cases}$ approches 0 from left

Begin by factoring the numerator and the denominator. If both the denominator and the numerator have a common factor that reduces, then we say that the rational function has a **hole** at the corresponding x of that factor.

$$f(x) = \frac{\cancel{(x-1)}(x+2)}{(x+1)\cancel{(x-1)}}$$

f has a hole @ $x=1$

Vertical Asymptotes - A rational function has vertical asymptotes where the denominator is zero (after any cancellation of common factors from the denominator and numerator)

$$f(x) = \frac{x+2}{x+1}$$

f has a VA at $x=-1$

Horizontal asymptotes

HA

Case 1: If the degree of the numerator is less than the degree of the denominator, the the horizontal asymptote is $y = 0$.

$$f(x) = \frac{x^2+x+1}{x^5+x^3+5}$$

$2<5$ so f has $\boxed{y=0}$ as the HA.

x axis

Case 2: If the degree of the numerator equals the degree of the denominator, then the horizontal asymptote is $y =$ the ratio of the leading coefficients.

$$f(x) = \frac{2x^3+x+1}{3x^3+2x}$$

f has HA at $y = \frac{2}{3}$

Case 3: Lastly, if the degree of the numerator is higher than the degree of the denominator, then the rational function has no horizontal asymptotes.

$$f(x) = \frac{2x^3+x+1}{3x^2+2x}$$

NO HA