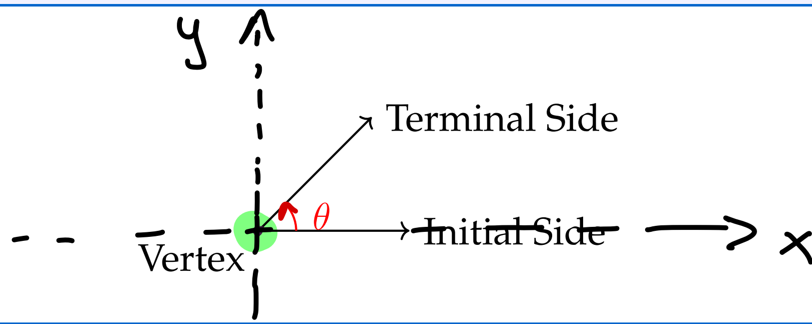
4.1 Radians and Degrees Measure and 4.2 Trig functions: The Unit Circle

1 Angles

1.1 Basic Definitions

Definition : Angle

An **angle** is formed by two rays that share a common endpoint. The common endpoint is called the **vertex** of the angle, one ray is called the **initial side**, and the other ray is called the **terminal side**.



Definition : Standard Position

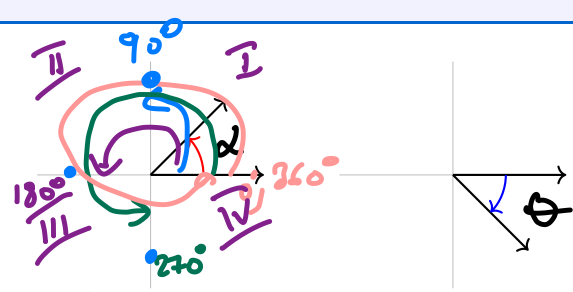
An angle is in **standard position** if:

- The vertex is at the origin of a rectangular coordinate system
- The initial side is along the positive x-axis

Notation: we use Greek letters to denote angles $\theta, \beta, \alpha, \gamma$

Definition : Positive and Negative Angles

- A **positive angle** is generated by a counterclockwise rotation from the initial side to the terminal side.
- A **negative angle** is generated by a clockwise rotation from the initial side to the terminal side.



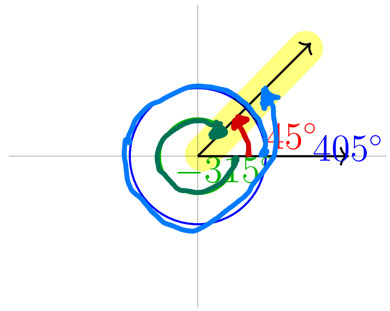
positive angle

negative angle

Definition : Coterminal Angles

Coterminal angles are angles in standard position that have the same terminal side.

If θ is an angle, then $\theta + 360°n$ where n is an integer, is coterminal with θ.



Coterminal Angles: $45°$, $405°$, and $-315°$

Example 1: Finding Coterminal Angles

Find two coterminal angles (one positive and one negative) for $\theta = 60°$.

Solution:

For a positive coterminal angle, add $360°$: $60° + 360° = 420°$

For a negative coterminal angle, subtract $360°$: $60° - 360° = -300°$

Another positive : $60° + 2 \cdot 360° = 780°$

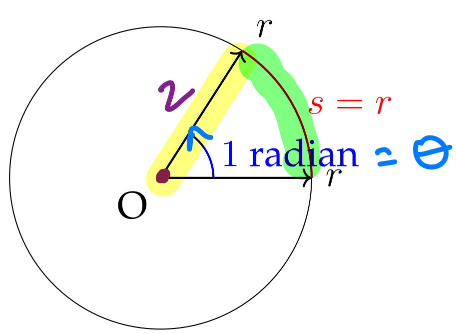
2 Radian Measure

2.1 Definition of Radian

Definition : Radian

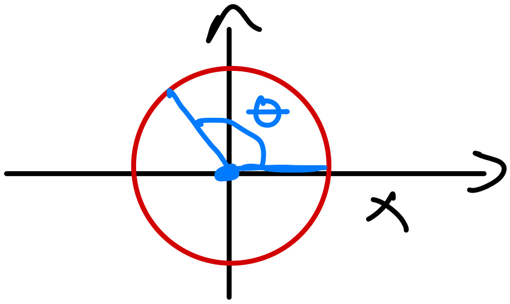
A **radian** is the measure of an angle that, when placed at the center of a circle, intercepts an arc equal in length to the radius of the circle.

$s =$ arc



Definition : Central Angle

A **central angle** is an angle whose vertex is at the center of a circle, with its sides being radii of the circle.



θ is a central angle

360° = 2π radians

3 Degree and Radian Conversions

3.1 Converting Between Degrees and Radians

Definition : Conversion Formulas

To convert between degrees and radians, use the following relationships:

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$
$$1^\circ = \frac{\pi}{180} \text{ radians} \approx 0.01745 \text{ radians}$$

Therefore:

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$
$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

$\frac{120}{180} = \frac{12}{18} = \frac{2}{3}$

Example 2: Converting from Degrees to Radians

Convert the following angles from degrees to radians in terms of π: a) 45° b) 120° c) -30°

Solution:

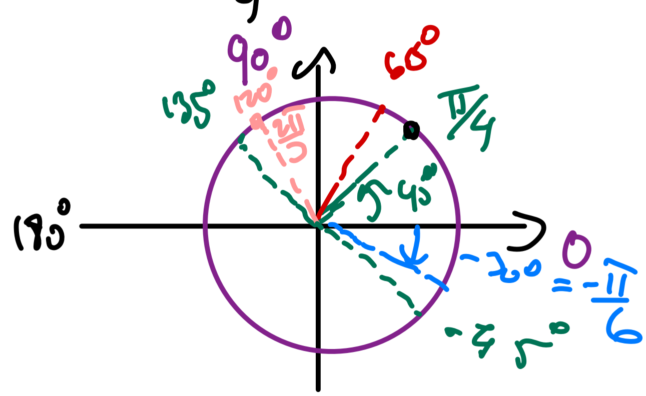
$45^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{4}$ 45° is $\frac{\pi}{4}$ radians

a) 45° in radians:

b) 120° in radians: $120^\circ \frac{\pi}{180^\circ} = \frac{2\pi}{3}$

c) -30° in radians: $-30^\circ \cdot \frac{\pi}{180^\circ} = -\frac{\pi}{6}$

$\cancel{45^\circ \cdot \frac{180^\circ}{\pi} = \text{radian}}$



Example 4: Finding Complements and Supplements

Find the complement and supplement of each angle: a) $25°$, b) $\frac{\pi}{4}$ radians, and c) $100°$

Solution:

a) For $\theta = 25°$:

Complement = $(90° - 25°) = 65°$
Supplement = $(180° - 25°) = 155°$

b) For $\theta = \frac{\pi}{4}$ radians:

Complement = $\pi/2 - \pi/4 = \pi/4$
Supplement = $\pi - \pi/4 = \frac{4\pi}{4} - \frac{\pi}{4} = 3\pi/4$

c) For $\theta = 100°$:

Complement = no complement
Supplement = $180° - 100° = 80°$

4.2 Arc Length and Area of a Sector

Definition : Arc Length and Area of a sector

If a central angle θ in a circle of radius r intercepts an arc, then the arc length s is given by:

$$s = r\theta$$

where θ must be measured in radians. The area A of a sector is given by:

$$A = \frac{1}{2}r^2\theta$$

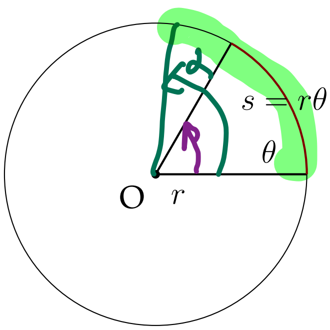


Figure 1: Arc length of a sector

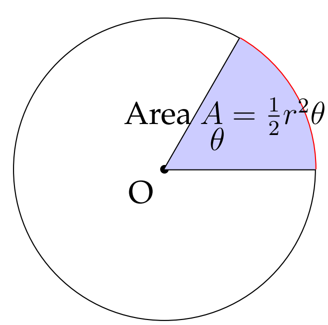


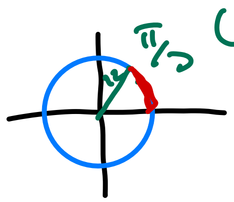
Figure 2: Area of a sector

Example 5: Finding Arc Length

Find the arc length intercepted by a central angle of $\frac{\pi}{3}$ radians in a circle with radius 12 cm.

θ $\qquad$ $r = 12\,cm$

Solution: Using the arc length formula with $r = 12$ cm and $\theta = \frac{\pi}{3}$ radians:



$$S = r\theta = 12cm \cdot \frac{\pi}{3} = 4\pi cm \simeq 12.57cm$$

Example 6: Finding Area of a Sector

Find the area of a sector formed by a central angle of 72° in a circle with radius 5 m.

$$A = \frac{1}{2} r^2 \theta \rightarrow \text{in radians}$$

72° convert in radians $\qquad 72^\circ \cdot \frac{\pi}{180^\circ} = \frac{72\pi}{180} = \frac{2\pi}{5}$ radians

$$A = \frac{1}{\cancel{2}} (5m)^{\cancel{2}} \left(\frac{\cancel{2}\pi}{\cancel{5}}\right) = 5\pi\, m^2 \approx 15.71\, m^2$$

~~900 m²~~

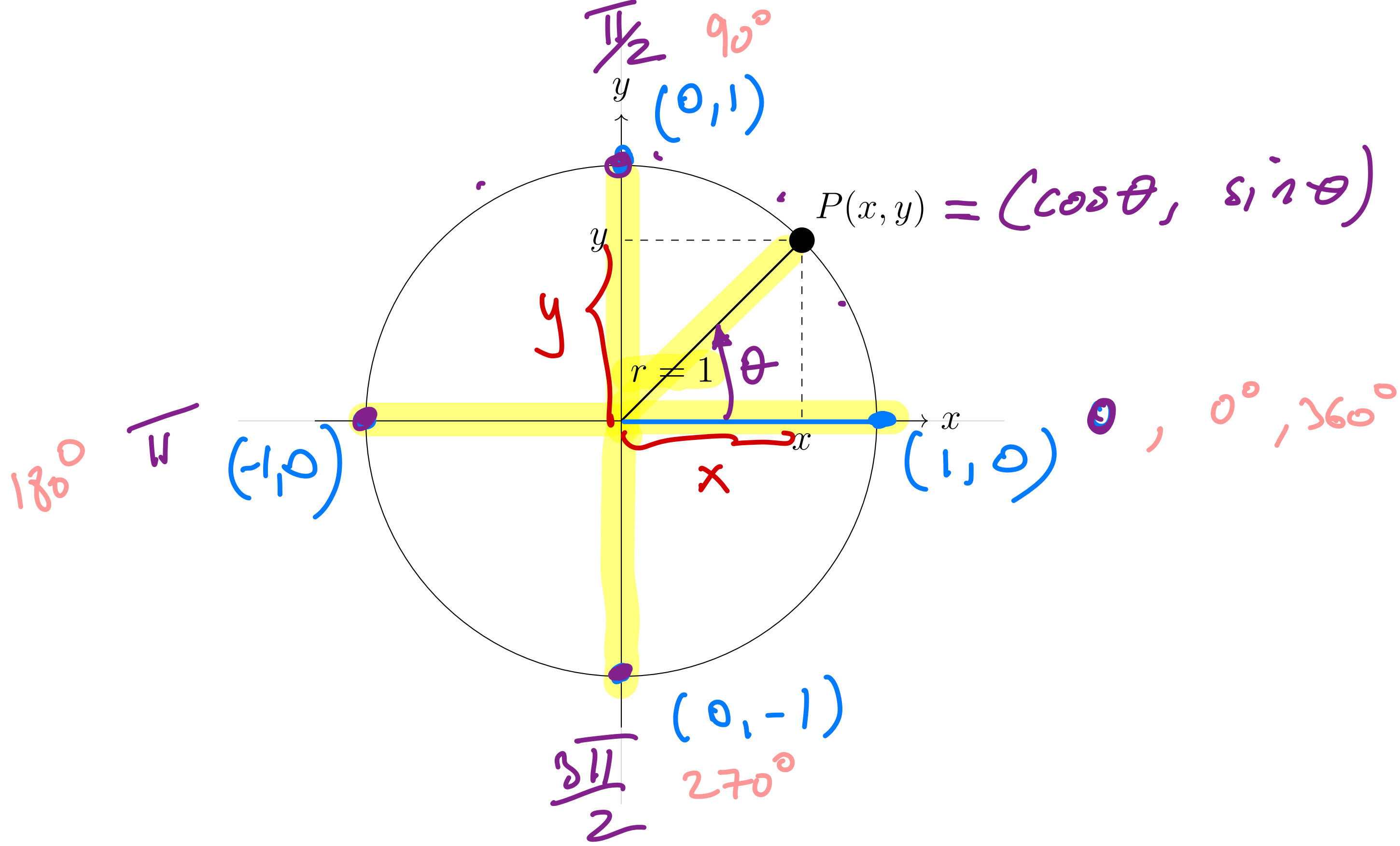
5 The Unit Circle and Trigonometric Functions

5.1 Definition of the Unit Circle

Definition : Unit Circle

The **unit circle** is a circle with radius 1 and center at the origin of the coordinate plane. Every point on the unit circle is at a distance of 1 unit from the origin.
The equation of the unit circle is:

$$x^2 + y^2 = 1$$



5.2 Six Trigonometric Functions: sine, cosine, tangent, cosecant, secant, cotangent

sin cos tan csc sec cot

Definition : Primary Trigonometric Functions

For a point $P(x, y)$ on the unit circle corresponding to an angle θ in standard position:

$$\sin\theta = y$$
$$\cos\theta = x$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}, \quad \cos\theta \neq 0$$

Definition : Reciprocal Trigonometric Functions

The reciprocal trigonometric functions are defined as:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}, \quad \sin\theta \neq 0$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}, \quad \cos\theta \neq 0$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}, \quad \sin\theta \neq 0$$

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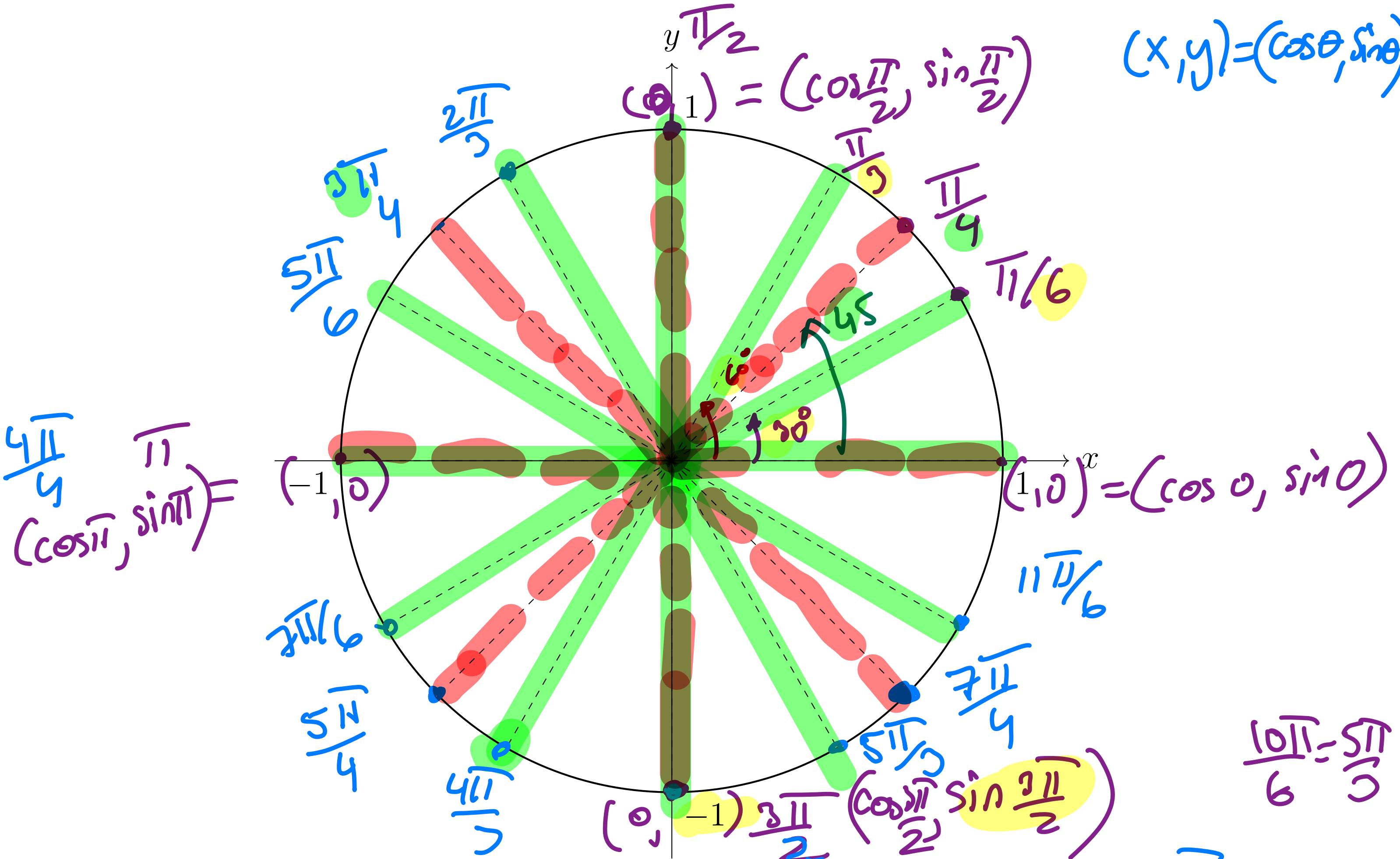

$\frac{8\pi}{6} = \frac{4\pi}{3}$

Unit Circle with Common Angles

$\frac{6\pi}{4} = \frac{3\pi}{2}$

5.3 Evaluating Trigonometric Functions

Example 7: Evaluating Trig Functions for Special Angles

Find the values of all six trigonometric functions for $\theta = 30^\circ$.

Solution:
For $\theta = 30^\circ = \frac{\pi}{6}$ radians, the point on the unit circle is:

$$P(\cos 30^\circ, \sin 30^\circ) = P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Therefore:
$\sin 30^\circ =$
$\cos 30^\circ =$
$\tan 30^\circ =$
$\csc 30^\circ =$
$\sec 30^\circ =$
$\cot 30^\circ =$