Day 10, Monday March 3rd

Coming up: WebAssig HW 7 due today

HW 8 due Wed

Written HW 2 was due 1pm today

Remember WebAssign HW due every Monday & Wednesday 15% of entire final grade

- Quiz on Polynomial Functions
- Exam 2 on Wednesday 3/12
- NO index card

Group Practice: Do @ home as written HW2

1. Divide the polynomial $f(x) = (x^3 - 9x^2 + 26x - 24)$ by $d(x) = (x - 2)$.

   (a) Does $x - 2$ divide evenly in $f(x)$? yes

   (b) Calculate the value of $f(2)$.

$$
\begin{array}{r|l}
 & x^2 - 7x + 12 \\
\hline
x-2 & x^3 - 9x^2 + 26x - 24 \\
 & x^3 - 2x^2 \\
\hline
 & -7x^2 + 26x - 24 \\
 & -7x^2 + 14x \\
\hline
 & 12x - 24 \\
 & 12x - 24 \\
\hline
 & 0
\end{array}
$$

$r = 0$

$f(2) = 2^3 - 9 \cdot 2^2 + 26 \cdot 2 - 24 = 8 - 36 + 52 - 24 = 0$

2. Divide $f(x) = (x^4 - 5x^3 + 2x^2 + 8x - 6)$ by $d(x) = (x - 3)$

   (a) Does $x - 3$ divide evenly in $f(x)$?

   (b) Calculate the value of $f(3)$.

$f(3) = -18$

$$
\begin{array}{r|l}
 & x^3 - 2x^2 - 4x - 4 \\
\hline
x-3 & x^4 - 5x^3 + 2x^2 + 8x - 6 \\
 & x^4 - 3x^3 \\
\hline
 & -2x^3 + 2x^2 + 8x - 6 \\
 & -2x^3 + 6x^2 \\
\hline
 & -4x^2 + 8x - 6 \\
 & -4x^2 + 12x \\
\hline
 & -4x - 6 \\
 & -4x + 12 \\
\hline
 & -18
\end{array}
$$

$\leftarrow r$

$x+2$, $f(-2) = ?$

The Remainder and Factor Theorem

$(x+a) = (x-(-a))$

Remainder Theorem:

If a polynomial $f(x)$ is divided by $x-a$, then the remainder is equal to $f(a)$.

Factor Theorem:

A polynomial $f(x)$ has a factor $\underbrace{(x-a)}_{\text{divides evenly}}$ if and only if $f(a)=0$.

Recall: Polynomials :

- factored form $f(x) = (x+1)(x-2)(x+3)^2$
- $f(x) = x^3 - 3x^2 + 4$, $(x-2)$ factor

  long division

  $f(x) = (x-2)(x^2-x-2)$

  $= (x-2)(x-2)(x+1)$
- Next $f(x) = 2x^3 + 3x^2 - 8x + 3$

The Rational Zero Test

What if we are not given any roots or zeros of the polynomial but we are still asked to factor it. For example, in Example 1 we were told to divide $f(x) = x^3 - 3x^2 + 4$ by $x - 2$ and in doing that we were able to completely factor out $f(x)$. Next, we are interested in how to find potential zeros. This is the result of The Rational zero Test.

real numbers

rational numbers : $1/2$, $\frac{7}{1}$, $1/3$
0.5 $\quad 0.\bar{3}$

irrational numbers : $\pi = 3.141\ldots$, $e = 2.72$, $\sqrt{2} = 1.41\ldots$

The Rational Zero Test

If the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q} \rightarrow \text{reduced form} \qquad \frac{\pm 2}{\pm 4} = \frac{\pm 1}{\pm 2}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and
q is a factor of the leading coefficient a_n.

Actual zeros : $-1, 2$
$f(-1)=0$ & $f(2)=0$

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coeff.}} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1} = \pm 1, \pm 2, \pm 4$$

$f(x) = 1x^3 - 3x + 4$

constant term is : 4

factors of 4 : $\pm 1, \pm 2, \pm 4$

leading coefficient : 1

factors : ± 1

Example 4: Find the rational zeros of

1. $f(x) = x^3 - 3x^2 + 4$.

see previous page

$\rightarrow$ leading coef $\rightarrow$ const term

2. $h(x) = 2x^3 + 3x^2 - 8x + 3$

factors of const term : $\pm 1, \pm 3$

factors of leading coef : $\pm 1, \pm 2$

possible rational zeros: $\dfrac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}$

From graph zeros (x int) : $-3, 1/2, 1$

$f(x) = (x+3)(2x-1)(x-1)$

$\hookrightarrow 2x - 1 = 0$

$x = 1/2$

$x - \frac{1}{2} = 2x - 1$

3. $g(x) = x^3 - 2$

const term : -2

leading coeff : 1

Possible rat zeros $= \dfrac{\pm 1, \pm 2}{\pm 1} = \pm 1, \pm 2$

$f(-1) = -1 - 2 = -3 \neq 0$

$f(1) = 1^3 - 2 = -1 \neq 0$

$f(2) = 2^3 - 2 = 6 \neq 0$

$f(-2) = (-2)^3 - 2 = -10 \neq 0$

Actual zeroes : is an irrational zero $\sqrt[3]{2}$

$x^3 - 2 = 0$

$x^3 = 2 \Rightarrow x = \sqrt[3]{2}$

$f(x) = (x - \sqrt[3]{2})(\underbrace{\qquad}_{\text{second degree om}})$

A degree 3 can have at most 3 real roots
x intercepts

A nth degree polynomial can have at most n real roots.
In the complex number system , every nth degree polynomial has exactly n zeros.

Recall Complex Numbers

$\boxed{\sqrt{-1} = i}$ or $\boxed{i^2 = -1}$

$\sqrt{-15} = \sqrt{(-1)(15)} = \sqrt{-1}\sqrt{15} = i\sqrt{15}$
$\sqrt{-9} = \sqrt{(-1)9} = \sqrt{-1}\sqrt{9} = i3 = 3i$

A complex number is a number of the form $a + bi$, where a and b are constants(real numbers)

a is called the real part

b is called the imaginary part.

EX $2 + 3i$ $a=2$, $b=3$, $2-3i$ is called the conjugate of $2+3i$

$\boxed{a+bi \text{ and } a-bi \text{ are conjugates}}$

EX $\sqrt{5}i = i\sqrt{5}$; $a=0$, $b=\sqrt{5}$
pure imaginary

EX -17 $a=-17$, $b=0$

Note: -17 is a **both** a complex number **and** a real number.
<u>**Note**</u> Every real number is a complex number, but not every complex number is a real number.

Types of Zeros of a polynomial function f

- Real $\begin{cases} \text{rational : } -3,\ 1/2,\ 2/3 & \text{ex: } f(x)=(x+3)(x-\frac{2}{3}) \\ \text{irrational : } \sqrt{2}, \sqrt[3]{2}, \pi & \text{ex: } f(x)=(x-\pi) \end{cases}$

  are x-intercepts
- Complex zeros are <u>not</u> x intercepts
  They come in pairs :
  If $1+2i$ is a zero , then $1-2i$ is also a zero.

If $x = 1+2i$ is a zero

then $(x-(1+2i))$

$x = 1-2i$ is a zero

$(x-(1-2i))$

$x^2+4=0$

$x^2=-4$

$x=\pm\sqrt{-4}$

$x=\pm 2i$

$(x-(1+2i))(x-(1-2i)) =$

$= (x-1-2i)(x-1+2i)$

$= x^2 - x + \cancel{x2i} - x + 1 - \cancel{2i} + \cancel{-x2i} + \cancel{2i} - 4i^2$

$-4(-1)$

$= x^2-2x+5$

Fundamental Theorem of Algebra

Every polynomial function with degree $n \geq 1$ can be factored into n linear factors of the following form:

$f(x) = a(x - r_1) \cdot (x - r_2) \cdot (x - r_3) \cdot ... \cdot (x - r_n)$

where $r_1, r_2, r_3, ..., r_n$ are complex numbers and $a \neq 0$.

Example 5: Completely factor the polynomial over the complex numbers.

$f(x) = (x - 1)(x^2 + 4)(x^2 - 3)$

$x^2 - 3 = 0$
$x^2 = 3$
$x = \pm\sqrt{3}$
$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$

$f(x) = (x-1)(x^2+4)(x-\sqrt{3})(x+\sqrt{3})$
factored completely over real number

$x^2 + 4 = 0$
$x^2 = -4$
$x = \pm\sqrt{-4}$
$x = \pm 2i$
$x^2 + 4 = (x - 2i)(x + 2i)$

$f(x) = (x-1)(x-\sqrt{3})(x+\sqrt{3})(x-2i)(x+2i)$
↳ factored completely over the complex number

Example 6: List all possible rational zeros. Find all zeros and completely factor f over the complex numbers.

$f(x) = 2x^4 + x^3 + 17x^2 + 9x - 9$

constant term : -9 w/ factors : $\pm 1, \pm 3, \pm 9$

lead coef 2 w/ factors : $\pm 1, \pm 2$

possible rational zeros are $= \dfrac{\pm 1, \pm 3, \pm 9}{\pm 1, \pm 2} = \pm 1, \pm 1/2, \pm 3, \pm 3/2, \pm 9, \pm 9/2$

Graph : 2 x intercept $x = -1$, $x = 1/2$

$\downarrow$ $(x+1)$ $\quad$ $\downarrow$ $(x - 1/2)$ or $(2x-1)$

To get the 2 remaining complex zeros we divide $f(x)$ by $(x+1)(2x-1) = 2x^2 + x - 1$

$$\begin{array}{r} x^2+9 \\ 2x^2-x-1 \overline{)\,2x^4+x^3+17x^2+9x-9} \end{array}$$

$$2x^4+x^3+17x^2+9x-9 = (x+1)(2x-1)(x^2+9)$$

$$= (x+1)(2x-1)(x-3i)(x+3i)$$

$$x^2+9=0$$

$$x^2=-9$$

$$x=\pm 3i$$

$$x^2-2x+5=0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)5}}{2 \cdot 1} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$