Mth 146: Day 8
7.8 Improper Integrals

1 Part 1: New Concepts & Worked Examples

$\int_a^b f(x)\,dx = \text{antiderivative}\Big|_a^b$

So far we have examined definite integrals only for continuous functions $f(x)$ over finite intervals $[a, b]$, since those are the conditions under which the Fundamental Theorem of Calculus applies.

Our next goal is to examine integrals

- over an unbounded interval : $[1, \infty)$ or $(-\infty, 5]$ or $(-\infty, \infty)$

$\int_1^\infty f(x)\,dx = ?$ or $\int_{-\infty}^\infty f(x)\,dx = ?$ or $\int_{-\infty}^{3} f(x)\,dx = ?$

where is still continuous. Type I improper integral.

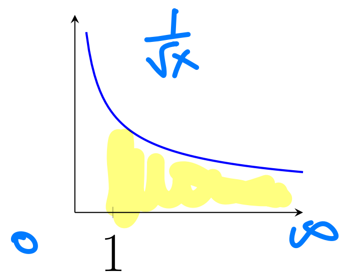
- where the integrand is an unbounded function (the function is discontinuous somewhere in the interval and it goes to ∞ or $-\infty$ at that point)

$\int_a^b f(x)\,dx$ where $f(x)$ is discontinuos somewhere in the interval $[a, b]$. Type II Improper Integrals

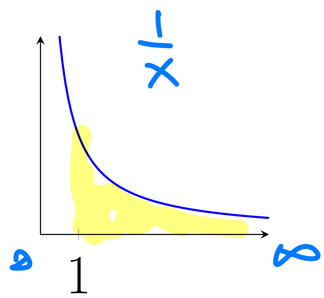
next class

These types of integral are called **improper**.

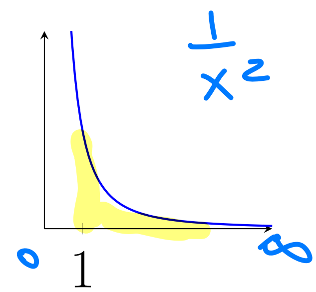
Here are the graphs of $\dfrac{1}{\sqrt{x}}$, $\dfrac{1}{x}$, and $\dfrac{1}{x^2}$ on $[1, \infty)$. As you look at each one, think about whether the area under the curve on this interval appears to be finite or infinite:



$\dfrac{1}{\sqrt{x}}$



$\dfrac{1}{x}$



$\dfrac{1}{x^2}$

Algebraically:

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \boxed{\int_1^{t} \frac{1}{\sqrt{x}}\,dx} = \lim_{t\to\infty} \frac{x^{-1/2+1}}{-1/2+1}\Bigg|_1^t$$

$$= \lim_{t\to\infty} 2\sqrt{x}\,\Big|_1^t = \lim_{t\to\infty}\left(2\sqrt{t} - 2\sqrt{1}\right) =$$

$$= 2\sqrt{\infty} - 2 = \infty - 2 = \infty$$

(the improper integral diverges)

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \underbrace{\int_1^{t} \frac{1}{x}\,dx} = \lim_{t\to\infty} \ln x\,\Big|_1^t$$

$$= \lim_{t\to\infty}\left(\ln t - \ln 1\right) = \ln\infty - \ln 1$$

$$= \infty - 0 = \infty$$

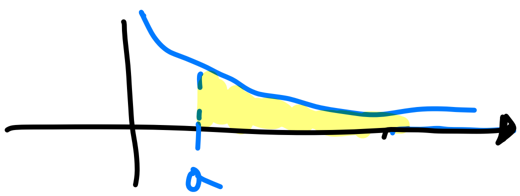
also diverges.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{t} + \frac{1}{1} = 0+1 = 1$$

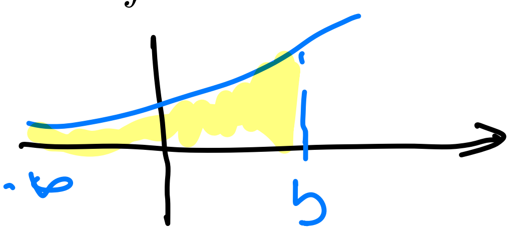
converges to 1

Suppose a and b are any real numbers and $f(x)$ is a function.

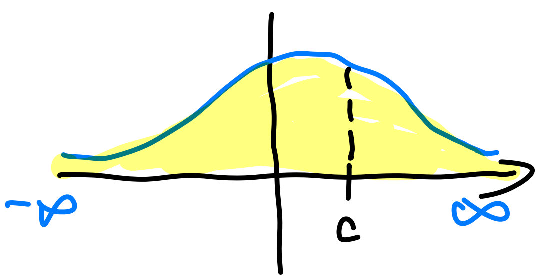
1. If f is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$



2. If f is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$



3. If f is continuous on $(-\infty, \infty)$, then



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

$$= \lim_{t\to-\infty} \int_t^c f(x)\,dx + \lim_{z\to\infty} \int_c^z f(x)\,dx$$

In all three cases, if the limits involved exist, then we say that the improper integral **converges** to the value determined by those limits, and if the limits are infinite or fail to exist, then we say that the improper integral **diverges**.

Example 1. $\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$, $\frac{1}{x} = \frac{1}{x^1}$, $\frac{1}{x^2}$

$p = 1/2$ $\quad$ $p = 1$ $\quad$ $p = 2$

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

For $p=1$ the $\int_1^\infty \frac{1}{x^1}\,dx$ diverges

For $p>1$. $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p}-1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{1-p}(-1) = \frac{1}{p-1}$$

For $p<1$ $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p}-1\right) = \infty$

The p-integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ when $p>1$ and diverges when $p\le 1$.

$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}$

Example 2. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_1^\infty x^{-3/2}\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx$$

$$= \frac{1}{p-1} = \frac{1}{1.5-1} = \frac{1}{.5} = 2$$

$p = 3/2 = 1.5$

$p>1$ so converges to

Example 3. Determine whether the improper integral converges and, if so, evaluate it.

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} \underbrace{\int_0^t xe^{-x^2}\,dx}_{} = \lim_{t\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}e^0\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{e^{t^2}}+\frac{1}{2}\right)$$

$\frac{1}{e^{t^2}} \to 0$

$$= \frac{1}{2}$$

On the side $\int xe^{-x^2}dx = -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^{-x^2}+C$

$u=-x^2$

$du=-2x\,dx$

$-\frac{1}{2}du = x\,dx$

Example 4. Determine whether the improper integral converges and, if so, evaluate it.

$\int_1^\infty \frac{\ln(x)}{x}\,dx$ check your answers:

$u = \ln x$

$du = \frac{1}{x}dx$

$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$

$= \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t$

$= \infty - 0$

$= \infty$ it diverges

Example 5. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$

$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{s\to\infty} \int_0^s \frac{1}{1+x^2}\,dx$

$= \lim_{t\to-\infty} \arctan x \Big|_t^0 + \lim_{s\to\infty} \arctan x \Big|_0^s$

$\arctan 0 - \arctan(-\infty) + \arctan(\infty) - \arctan 0$

$\arctan 0 \to 0$, $\arctan 0 \to 0$

$\tan ? = 0$

$\frac{\sin}{\cos}$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Try the following individually or in small groups. The final answer is given as a footnote at the bottom of the page — only the final answer, not the steps, so use it to check your work rather than replace it.

1. $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx.$[1] $\displaystyle = \lim_{t\to-\infty}\int_t^{-2} \frac{2}{x^2-1}\,dx$

$$\circledast = \lim_{t\to-\infty}\left(\int_t^{-2}\frac{1}{x-1}\,dx - \int_t^{-2}\frac{1}{x+1}\,dx\right)$$

$$= \lim_{t\to-\infty} \ln|x-1|\Big|_t^{-2} - \ln|x+1|\Big|_t^{-2}$$

$$= \lim_{t\to-\infty} \ln 3 - \ln|t-1| - \ln 1 + \ln|t+1|$$

$$= \ln 3 - \lim_{t\to\infty} \ln\left|\frac{t-1}{t+1}\right| = \ln 3 - \ln \lim_{t\to\infty}\left|\frac{t-1}{t+1}\right| = \ln 3$$

$\to 1$ & $\ln 1 = 0$

$$\circledast\ \frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$2 = Ax + A + Bx - B$

$A + B = 0 \qquad A = -B \qquad A = 1$

$A - B = 2 \qquad -2B = 2$

$B = -1$

[1]Answer: The integral converges to $\ln 3$.

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Define and compute (if possible) integrals of the form**

$$\int_a^\infty f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx, \qquad \int_{-\infty}^\infty f(x)\,dx.$$

Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

Reflection:
