

Example 6. $\int (3e^x - \sec^2 x) dx$

$$\begin{aligned} &= \int 3e^x dx - \int \sec^2 x dx \\ &= 3e^x - \tan x + C \end{aligned}$$

Layer 2: Manipulating into Elementary Form

Sometimes the integrand isn't elementary as written, but algebra or a trig identity can rewrite it into pieces that are.

Common Moves

- **Expand** products or powers: $(x + 1)^2 = x^2 + 2x + 1$
- **Split** a fraction with one denominator into separate terms: $\frac{x^2 + 3}{x} = x + \frac{3}{x}$
- **Rewrite** radicals and reciprocals as powers of x : $\sqrt{x} = x^{1/2}$, $\frac{1}{x^3} = x^{-3}$
- **Use trig identities** to replace a non-elementary trig expression:
 $\tan^2 x = \sec^2 x - 1$, $\sin^2 x = \frac{1 - \cos 2x}{2}$
- **Polynomial long division** when the degree of the numerator is $\geq$ the degree of the denominator: rewrite $\frac{p(x)}{q(x)}$ as (quotient) + $\frac{\text{remainder}}{q(x)}$

Example 7. $\int \frac{x^3 - 4x}{x^2} dx = \int \frac{x^3}{x^2} dx - \int \frac{4x}{x^2} dx$

$$\begin{aligned} &= \int x dx - 4 \int \frac{1}{x} dx \\ &= \frac{x^2}{2} - 4 \ln |x| + C \end{aligned}$$

Example 8. $\int \sqrt{x}(x-3) dx = \int x\sqrt{x} - 3\sqrt{x} dx$

$$\begin{aligned}
 &= \int x^{1/2+1} dx - 3 \int x^{1/2} dx \\
 &= \int x^{3/2} dx - 3 \int x^{1/2} dx \\
 &= \frac{2}{5} x^{5/2} - 3 \cdot \frac{2}{3} x^{3/2} + C
 \end{aligned}$$

$$\begin{aligned}
 \int x^n dx &= \frac{x^{n+1}}{n+1}
 \end{aligned}$$

$$\begin{aligned}
 \int x^{3/2} dx &= \frac{x^{3/2+1}}{3/2+1} \\
 &= \frac{x^{5/2}}{5/2}
 \end{aligned}$$

Example 9. $\int \tan^2 x dx = \int \sec^2 x - 1 dx$

$$\begin{aligned}
 &= \int \sec^2 x dx - \int 1 dx \\
 &= \tan x - x + C
 \end{aligned}$$

used
 $\tan^2 x = \sec^2 x - 1$

Layer 3: u -Substitution

$$\begin{aligned}
 f(x) &= \cos x \\
 g(x) &= x^2
 \end{aligned}$$

$$f(g(x)) = \cos x^2$$

If the integrand is a composition $f(g(x))$ and the derivative of the “inside function” $g(x)$ also shows up (up to a constant multiple), substitution turns it into an elementary integral.

u -Substitution Formula:

$$\int f(g(x)) g'(x) dx = \int f(u) du, \quad u = g(x)$$

Steps:

1. Choose $u = g(x)$ — the inside piece: something raised to a power, inside a root/exponential/log, or in a denominator.
2. Compute $du = g'(x) dx$ and solve for the piece you need.
3. Rewrite the *entire* integral in terms of u only.
4. Integrate with respect to u , then substitute $g(x)$ back in for u .

Example 10. $\int 2x \cos(x^2) dx$

$$\begin{aligned}
 u &= x^2 \\
 du &= 2x dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \cos u du \\
 &= \sin u + C = \sin x^2 + C
 \end{aligned}$$

Example 11. $\int \frac{x}{x^2+1} dx$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + C$$

$$= \frac{1}{2} \ln |x^2+1| + C = \frac{1}{2} \ln(x^2+1) + C$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

Example 12. $\int \frac{x^3 + 3x + 1}{x^2 + 1} dx$

$$= \int x dx + \int \frac{2x+1}{x^2+1} dx$$

$$= \frac{x^2}{2} + \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$$

$$= \frac{x^2}{2} + \ln(x^2+1) + \arctan x + C$$

$$\begin{array}{r} x^2+1 \overline{) x^3+3x+1} \\ \underline{x^3+x} \\ 0+2x+1 \end{array}$$

$$\frac{x^3+3x+1}{x^2+1} = x + \frac{2x+1}{x^2+1}$$

~~Try $u = \sin x$~~

~~$$du = \cos x dx$$~~

Example 13. $\int \frac{\sin x}{\cos^3 x} dx$

$$= - \int \frac{1}{u^3} du$$

$$= - \int u^{-3} du$$

$$= - \frac{u^{-2}}{-2} + C = \frac{1}{2} \frac{1}{\cos^2 x} + C = \frac{1}{2} \sec^2 x + C$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

$$= - \int \frac{1}{u} du = - \ln | \cos x | + C$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$-du = \sin x dx$$

$$= \ln \left| \frac{1}{\cos x} \right| + C = \ln |\sec x| + C$$

Sometimes after substituting, a leftover x still remains in the integrand. If so, solve $u = g(x)$ for x and substitute that expression in as well before integrating.

Example 14. $\int (1+x)\sqrt{2-x} dx$

$$= \int (2-u)\sqrt{u} du$$

$$= \int 2\sqrt{u} - u\sqrt{u} du$$

$$= -3 \int u^{1/2} du + \int u^{3/2} du$$

$$= -3 \frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + C$$

$$= -2(2-x)^{3/2} + \frac{2}{5} \cdot (2-x)^{5/2} + C$$

$$u = 2-x$$

$$du = -dx$$

$$-du = dx$$

$$u + 3 = 2-x + 3$$

→ solve for x

$$u + x = 2$$

$$-u \quad -u$$

$$x = 2-u$$

$$1+x = 1+(2-u)$$

$$1+x = 3-u$$