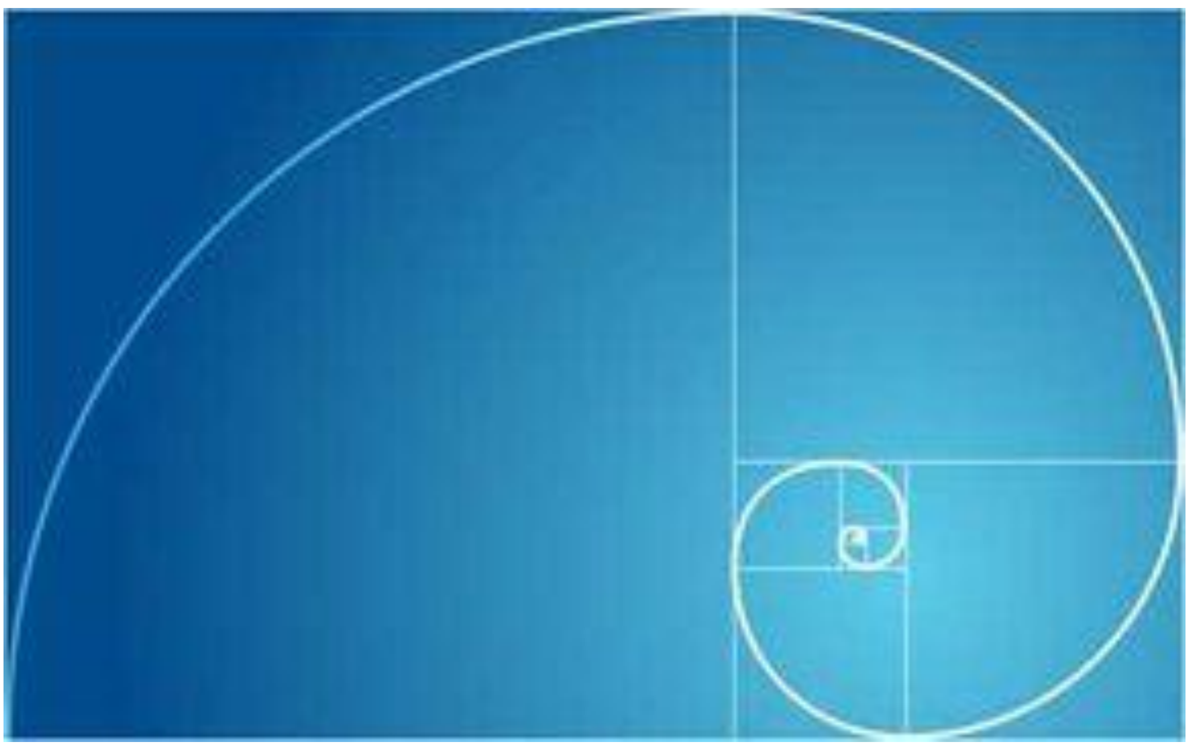
Contemporary Mathematics

Custom Text for the College of Lake County



This text consists of chapters selected from the texts:

Math in Society by David Lippman

College Mathematics for Everyday Life by Inigo, Jameson, Kozak, Lanzetta and Sonier

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CHAPTER 1
GRAPH THEORY

Excerpted from *Math in Society*, by David Lippman

Graph Theory

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

Sec. 1.1 Graphs: Basic Concepts and Definitions

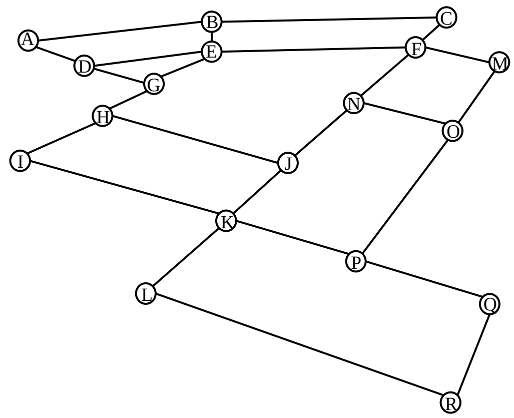
Example 1

Here is a portion of a housing development from Missoula, Montana[1]. As part of her job, the development's lawn inspector has to walk down every street in the development making sure homeowners' landscaping conforms to the community requirements.

[1] Sam Beebe. http://www.flickr.com/photos/sbeebe/2850476641/, CC-BY

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without having to do any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity.

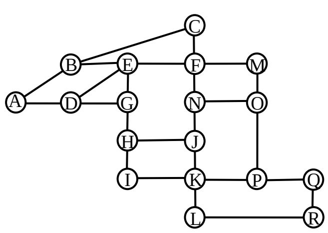
To do that, we first need to simplify the picture into a form that is easier to work with. We can do that by drawing a simple line for each street. Where streets intersect, we will place a dot.

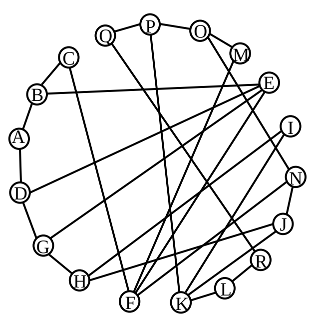


This type of simplified picture is called a **graph**.

> **Graphs, Vertices, and Edges**
> A **graph** consists of a set of dots or circles, called **vertices**, and a set of **edges** connecting pairs of vertices.

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph. Both of the graphs below are equivalent to the one drawn above since they show the same edge connections between the same vertices as the original graph.

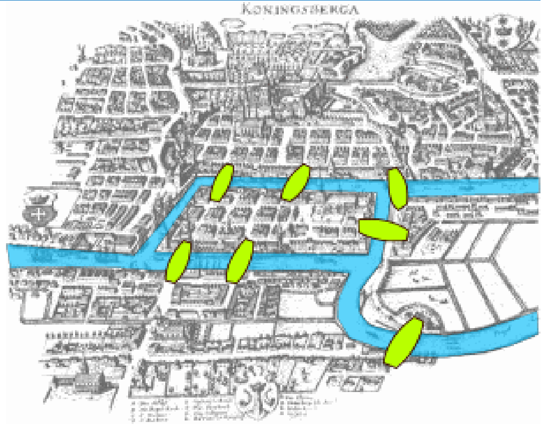




You probably already noticed that we are using the term *graph* differently than you may have used the term in the past to describe the graph of a mathematical function.

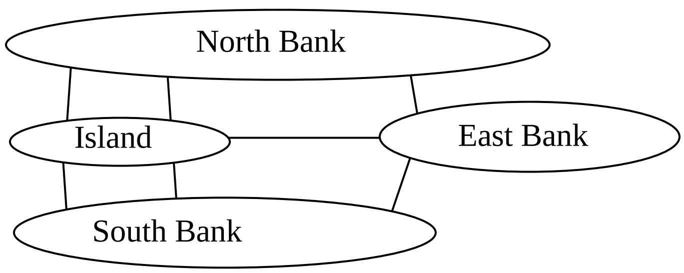
Example 2

Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[2].

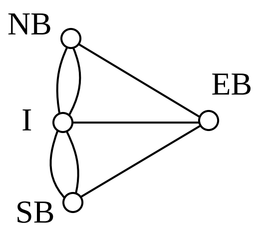


As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

While we haven't answered the actual question yet of whether or not there is a route which crosses every bridge once and returns to the starting location, the graph provides the foundation for exploring this question.

[2] Bogdan Giuşcă. http://en.wikipedia.org/wiki/File:Konigsberg_bridges.png

Definitions

While we loosely defined some terminology earlier, we now will try to be more specific.

Vertex

A vertex is a dot (or small circle) in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass – the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

Edges

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or that a route connecting the two locations exists, like direct airline flight. We often will describe an edge as AB, where A and B are the endpoints.

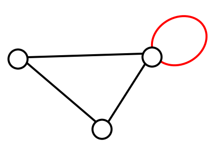
Adjacency

We say that two vertices A and B are adjacent if there is an edge connecting them.

Two edges e and f are adjacent if they have a common endpoint.

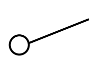
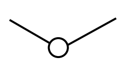
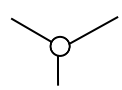
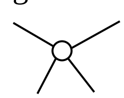
Loop

A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



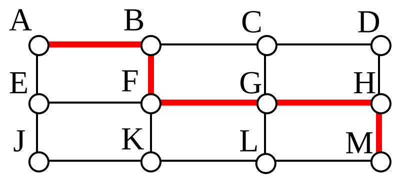
Degree of a vertex

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree2	Degree 3	Degree 4
				

Path

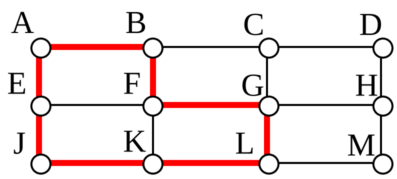
A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possiblExe paths in this graph.



We often denote a path as a sequence of adjacent vertices. For example, the path shown above can be written as ABFGHM.

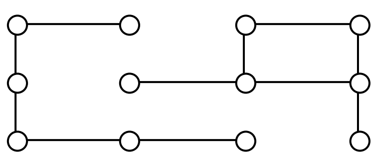
Circuit

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



Connected

A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is **disconnected**; there is no way to get from the vertices on the left to the vertices on the right.

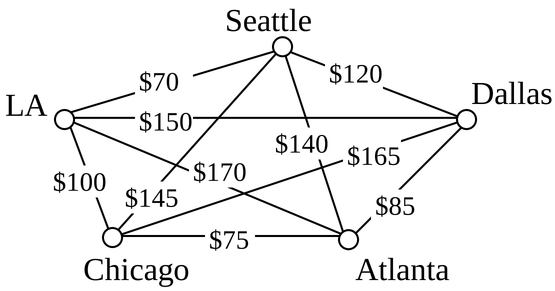


Weighted Graphs

Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Try it Now 1

1. The graph below shows 5 cities. The weights on the edges represent the airfare for a one-way flight between the cities.
   a. How many vertices and edges does the graph have?
   b. Is the graph connected?
   c. What is the degree of the vertex representing LA?
   d. If you fly from Seattle to Dallas to Atlanta, is that a path or a circuit?
   e. If you fly from LA to Chicago to Dallas to LA, is that a path or a circuit?



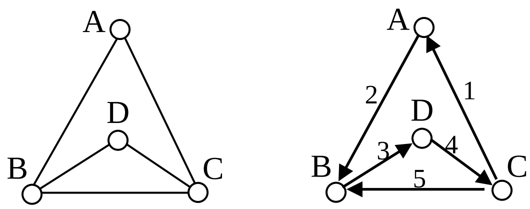
Sec. 1.2 Euler Circuits and the Chinese Postman Problem

In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Path
An **Euler path** is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.

Example 3

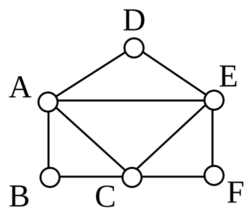
In the graph shown below, there are several Euler paths. One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

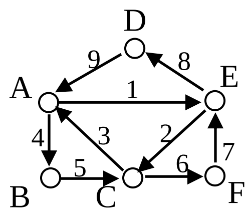


Euler Circuit
An **Euler circuit** is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example 4

The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.





Look back at the example used for Euler paths – does that graph have an Euler circuit? A few tries will tell you no; that graph does not have an Euler circuit. When we were working with shortest paths, we were interested in the optimal path. With Euler paths and circuits, we're primarily interested in whether an Euler path or circuit *exists*.

Why do we care if an Euler circuit exists? Think back to our housing development lawn inspector from the beginning of the chapter. The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit!

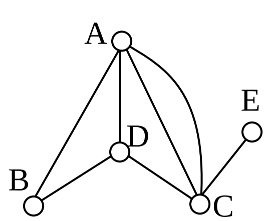
Luckily, Euler solved the question of whether a graph has an Euler circuit. His key insight was that if such a circuit exists, then every time the circuit visited a vertex, it must also exit the vertex. This tells us that as we travel the circuit, each time we visit a vertex, we "use up" two edges, one going in and one going out. And we can continue doing this as long as there are no vertices of odd degree. If there is a vertex of odd degree, then we would go in, but would then have to repeat an edge to go out. This gives us the following result:

Euler's Circuit Theorem

A graph will contain an Euler circuit if it is connected and all vertices have even degree.

Example 5

In the graph below, vertices A and C have degree 4, since there are 4 edges leading into each vertex. B is degree 2, D is degree 3, and E is degree 1. This graph contains two vertices with odd degree (D and E) and three vertices with even degree (A, B, and C), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



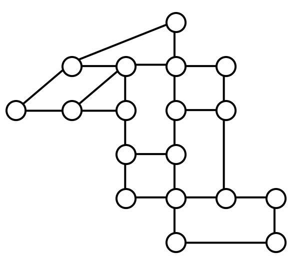
We can use similar reasoning to develop a criteria for whether a graph has an Euler path. If such a path exists, then each time we visit a vertex we would again use up two edges; the only exceptions would be the starting vertex and ending vertex, which would necessarily have odd degree. And if there are more than two vertices with odd degree, then the graph cannot have an Euler path. This gives the following:

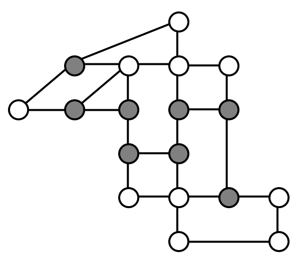
Euler's Path Theorem
A graph will contain an Euler path if it is connected and has exactly two vertices of odd degree.

Moreover, the path will start at one of the odd vertices and end at the other.

Example 6

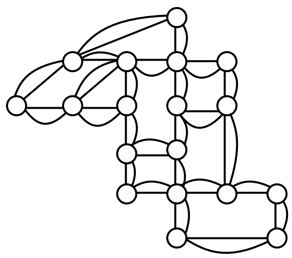
Is there an Euler circuit or an Euler path on the housing development lawn inspector graph we created earlier in the chapter? All the highlighted vertices have odd degree. Since there are more than two vertices with odd degree, there are no Euler paths or Euler circuits on this graph. Unfortunately, our lawn inspector will need to do some backtracking.





Example 7

When it snows in the same housing development, the snowplow has to plow both sides of every street. For simplicity, we'll assume the plow is out early enough that it can ignore traffic laws and drive down either side of the street in either direction. This can be visualized in the graph by drawing two edges for each street, representing the two sides of the street.



Notice that every vertex in this graph has even degree, so this graph does have an Euler circuit.

We now know how to determine if a graph has an Euler circuit; but if it does, how do we find one? For small graphs, it is usually possible to find an Euler circuit just by pulling out your pencil and trying to find one. But for larger, more complex graphs, we need an *algorithm*. An algorithm is a step-by-step procedure for solving a problem. These allow us to solve problems in a systematic way (and also allow for the use of computers). We will study a number of algorithms in this chapter.

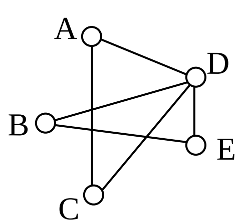
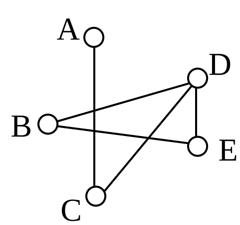
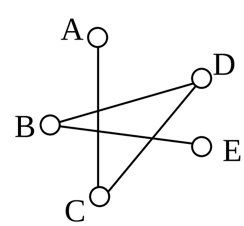
There is such a step-by-step procedure for finding Euler circuits, called **Fleury's algorithm.**

Fleury's Algorithm

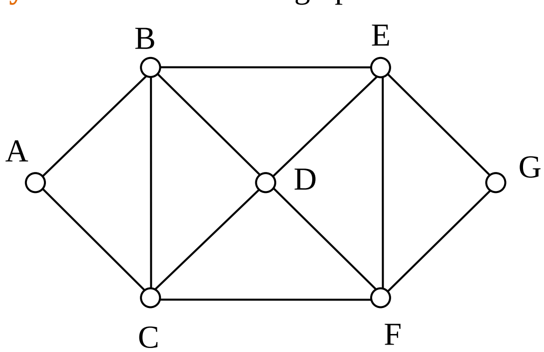
1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

Example 8

Let's find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex A.

Original Graph. Choosing edge AD.  Circuit so far: AD	AD deleted. D is current. Can't choose DC since that would disconnect graph. Choosing DE  Circuit so far: ADE	E is current. From here, there is only one option, so the rest of the circuit is determined.  Circuit: ADEBDCA

Try it Now 2: Does the graph below have an Euler Circuit? If so, find one.



Eulerization and the Chinese Postman Problem

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. In order to do that, she will have to duplicate some edges in the graph until an Euler circuit exists; more precisely, we want to add edges to connect vertices of odd degree. This process is called *eulerization*, and uses another of Euler's discoveries – namely, that for any graph, there must be an even number of vertices of odd degree:

Euler's Sum of Degrees Theorem
The sum of the degrees of all vertices in a graph is equal to twice the number of edges. (So the sum of the degrees of all vertices must be an even number.)

Thus, a graph always has an even number of vertices of odd degree.

The reasoning behind this theorem is fairly simple. Let AB be any edge. Then this edge contributes once to the degree of vertex A, and also contributes once to the degree of B. Thus the edge AB adds a total of two to the sum of degrees. As a result, the sum of degrees for all vertices must be double the number of edges. And if there were an odd number of odd degrees, then the total sum would be odd as well.

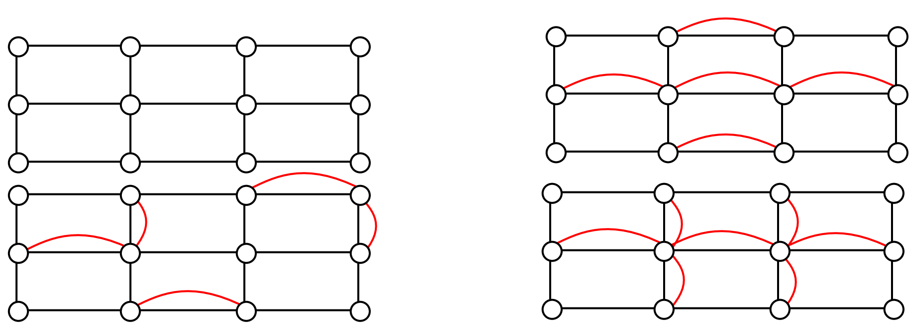
This theorem tells us that for any graph, the vertices of odd degree in a graph can be paired up. This is essential for the eulerization process:

Eulerization
Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degree. Connecting two odd degree vertices increases the degree of each, giving them both even degree. When two odd degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

It is important to note that we can only duplicate edges, not create edges where there wasn't one before. Duplicating edges would mean walking or driving down a road twice, while creating an edge where there wasn't one before is akin to installing a new road!

Example 9

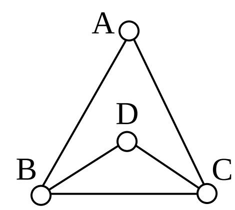
For the rectangular graph shown, three possible eulerizations are shown. Notice in each of these cases the vertices that started with odd degrees have even degrees after eulerization, allowing for an Euler circuit.



In the example above, you'll notice that the last eulerization required duplicating seven edges, while the first two only required duplicating five edges. For many applications (e.g. if we were eulerizing the graph to find a walking path), we would want the eulerization with minimal duplications. If the edges had weights representing distances or costs, then we would want to select the eulerization with the minimal total added weight.

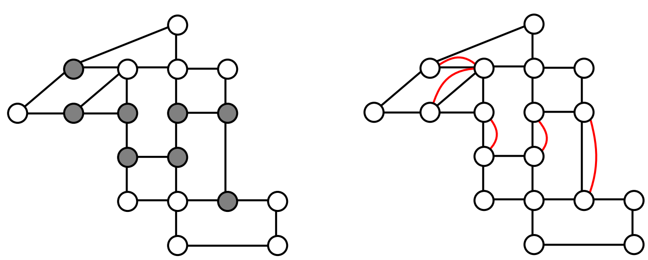
Try it Now 3

Eulerize the graph shown, then find an Euler circuit on the eulerized graph.



Example 10

Looking again at the graph for our lawn inspector from Examples 1 and 8, the vertices with odd degree are shown highlighted. With eight vertices, we will always have to duplicate at least four edges. In this case, we need to duplicate five edges since two odd degree vertices are not directly connected. Without weights we can't be certain this is the eulerization that minimizes walking distance, but it looks pretty good.



The problem of finding the optimal eulerization is called the Chinese Postman Problem, a name given by an American in honor of the Chinese mathematician Mei-Ko Kwan who first studied the problem in 1962 while trying to find optimal delivery routes for postal carriers. This problem is important in determining efficient routes for garbage trucks, school buses, parking meter checkers, street sweepers, and more.

Unfortunately, algorithms to solve this problem are fairly complex. Some simpler cases are considered in the exercises.

Sec. 1.3 Hamiltonian Circuits and the Traveling Salesman Problem

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver instead needs to visit every one of a set of delivery locations. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once.

Hamiltonian Circuits and Paths

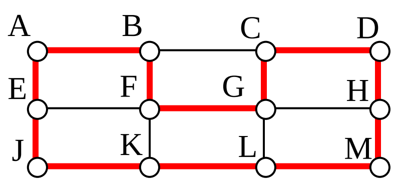
A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

Hamiltonian circuits are named for William Rowan Hamilton who studied them in the 1800's.

Example 11

One Hamiltonian circuit is shown on the graph below. There are several other Hamiltonian circuits possible on this graph. Notice that the circuit only has to visit every vertex once; it does not need to use every edge.

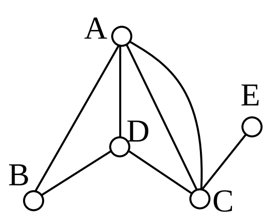
This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA. Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.



Unlike with Euler circuits, there is no nice theorem that allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs.[3] However, there is one observation that is quite useful in narrowing our search. If there is a Hamiltonian circuit, then that circuit must place exactly two edges on each vertex. This means that if there is a vertex of degree 2, then *both* of the edges that have this vertex as an endpoint must be used in any Hamiltonian circuit. Moreover, if there is a vertex of degree 1, then a Hamiltonian circuit is impossible (although there still may be a Hamiltonian path).

Example 12

Does a Hamiltonian path or circuit exist on the graph below?

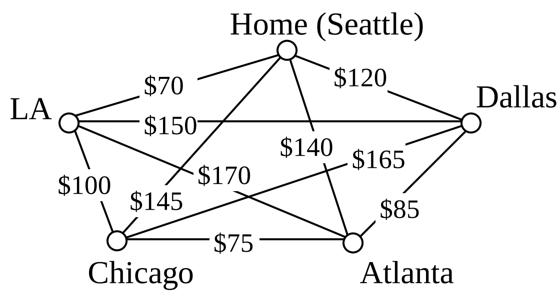


This graph cannot have a Hamiltonian circuit, because deg(E) = 1. From the picture, we see that once we travel to vertex E there is no way to leave without returning to C, so there is no possibility of a Hamiltonian circuit. However, if we start at vertex E we can find several Hamiltonian paths, such as ECDAB and ECABD.

[3] There are some theorems that can be used in specific circumstances, such as Dirac's theorem, which says that a Hamiltonian circuit must exist on a graph with *n* vertices if each vertex has degree *n*/2 or greater.

With Hamiltonian circuits, our focus will not be on existence, but on the question of optimization. That is, if we are given a graph where the edges have weights, how can we find the optimal Hamiltonian circuit, the one with lowest total weight?

This problem is called the **Traveling salesman problem** (TSP) because the question can be framed like this: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once then return home with the lowest cost?

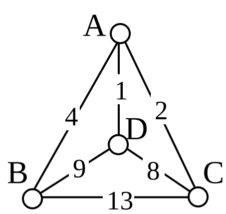


To answer this question of how to find the lowest cost Hamiltonian circuit, we will consider some possible approaches. The first option that might come to mind is to just try all different possible circuits:

Brute Force Algorithm (a.k.a. exhaustive search)
1. List all possible Hamiltonian circuits
2. Find the length of each circuit by adding the edge weights
3. Select the circuit with minimal total weight.

Example 13

Apply the Brute force algorithm to find the minimum cost Hamiltonian circuit on the graph below.



To apply the Brute force algorithm, we list all possible Hamiltonian circuits and calculate their weight:

Circuit	Weight
ABCDA	4+13+8+1 = 26
ABDCA	4+9+8+2 = 23
ACBDA	2+13+9+1 = 25

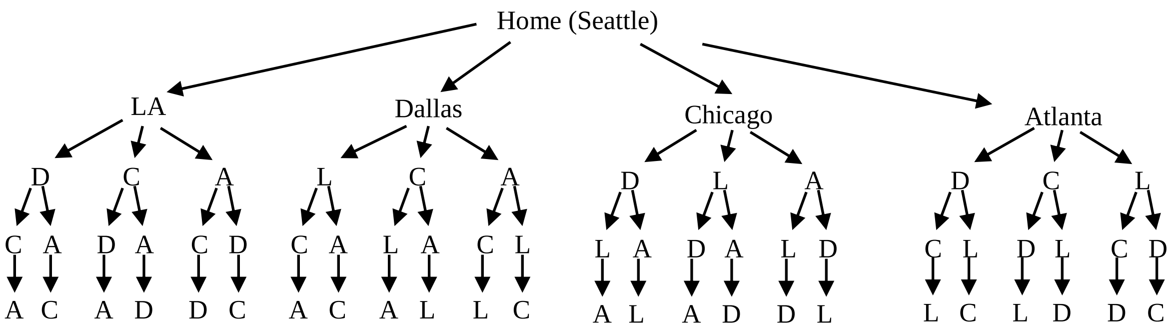
Note: These are the unique circuits on this graph. All other possible circuits are the reverse of the listed ones or start at a different vertex, but result in the same weights.

From this we determine that the second circuit, ABDCA, is the optimal circuit.

The Brute force algorithm is optimal; it will always produce the Hamiltonian circuit with minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph.**

Suppose we had a complete graph with five vertices like the air travel graph above. From Seattle there are four cities we can visit first. From each of those, there are three choices. From each of those cities, there are two possible cities to visit next. There is then only one choice for the last city before returning home.

This can be shown visually:



Counting the number of routes, we can see there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ routes. For six cities there would be $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ routes.

Number of Possible Circuits

For a complete graph with *n* vertices, there will be a total of

$$(n-1)! = (n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1$$

Hamiltonian circuits.

Half of these are duplicates in reverse order, so there are $\frac{(n-1)!}{2}$ unique circuits.

The exclamation symbol, !, is read "factorial" and is shorthand for the product shown.

Example 14

How many circuits would a complete graph with 8 vertices have?

A complete graph with 8 vertices would have $(8-1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ = 5040 possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique routes.

While this is a lot, it doesn't seem unreasonably huge. But consider what happens as the number of cities increase:

Cities	Unique Hamiltonian Circuits
9	8!/2 = 20,160
10	9!/2 = 181,440
11	10!/2 = 1,814,400
15	14!/2 = 43,589,145,600
20	19!/2 = 60,822,550,204,416,000

As you can see the number of circuits is growing extremely quickly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly, Brute Force is **not** an efficient algorithm.

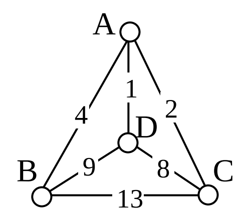
Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since it is not practical to use brute force to solve the problem, we turn instead to **heuristic algorithms**; efficient algorithms that give approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

Nearest Neighbor Algorithm (NNA)

1. Select a starting point.
2. Move to the nearest unvisited vertex (the edge with smallest weight).
3. Repeat until the circuit is complete.

Example 15

Consider our earlier graph, shown to the right.
Starting at vertex A, the nearest neighbor is vertex D with a weight of 1.
From D, the nearest neighbor is C, with a weight of 8.
From C, our only option is to move to vertex B, the only unvisited vertex, with a cost of 13.
From B we return to A with a weight of 4.

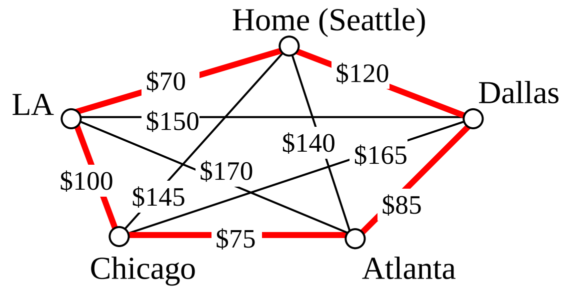


The resulting circuit is ADCBA with a total weight of 1+8+13+4 = 26.

We ended up finding the worst circuit in the graph! What happened? Unfortunately, while it is very easy to implement, the NNA is a **greedy** algorithm, meaning it only looks at the immediate decision without considering the consequences in the future. In this case, following the edge AD forced us to use the very expensive edge BC later.

Example 16

Consider again our salesman. Starting in Seattle, the nearest neighbor (cheapest flight) is to LA, at a cost of $70. From there:



LA to Chicago: $100
Chicago to Atlanta: $75
Atlanta to Dallas: $85
Dallas to Seattle: $120
Total cost: $450

In this case, the nearest neighbor algorithm did find the optimal circuit.

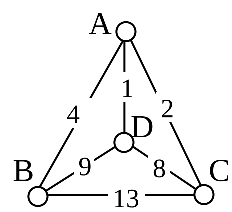
Going back to our first example, how could we improve the outcome? One option would be to redo the nearest neighbor algorithm with a different starting point to see if the result changed. Since nearest neighbor is so fast, doing it several times isn't a big deal.

Repeated Nearest Neighbor Algorithm (RNNA)

1. Do the Nearest Neighbor Algorithm starting at each vertex
2. Choose the circuit produced with minimal total weight

Example 17

We will revisit the graph from Example 17.



Starting at vertex A resulted in a circuit with weight 26.

Starting at vertex B, the nearest neighbor circuit is BADCB with a weight of 4+1+8+13 = 26. This is the same circuit we found starting at vertex A. No better.

Starting at vertex C, the nearest neighbor circuit is CADBC with a weight of 2+1+9+13 = 25. Better!

Starting at vertex D, the nearest neighbor circuit is DACBA. Notice that this is actually the same circuit we found starting at C, just written with a different starting vertex.

The RNNA was able to produce a slightly better circuit with a weight of 25, but still not the optimal circuit in this case. Notice that even though we found the circuit by starting at vertex C, we could still write the circuit starting at A: ADBCA or ACBDA.

Try it Now 4

The table below shows the time, in milliseconds, it takes to send a packet of data between computers on a network. If data needed to be sent in sequence to each computer, then notification needed to come back to the original computer, we would be solving the TSP. The computers are labeled A-F for convenience.

	A	B	C	D	E	F
A	--	44	34	12	40	41
B	44	--	31	43	24	50
C	34	31	--	20	39	27
D	12	43	20	--	11	17
E	40	24	39	11	--	42
F	41	50	27	17	42	--

a. Find the circuit generated by the NNA starting at vertex B.
b. Find the circuit generated by the RNNA.

While certainly better than the basic NNA, unfortunately, the RNNA is still greedy and will produce very bad results for some graphs. As an alternative, our next approach will step back and look at the “big picture” – it will select first the edges that are shortest, and then fill in the gaps.

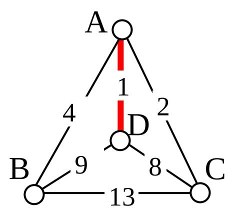
Cheapest Link Algorithm (a.k.a. Sorted Edges Algorithm)

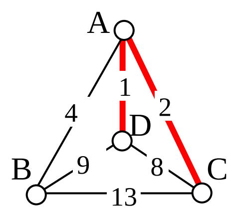
1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge to the circuit, unless:
    a. adding the edge would create a circuit that doesn’t contain all vertices, or
    b. adding the edge would give a vertex degree 3.
3. Repeat until a circuit containing all vertices is formed.

Example 18

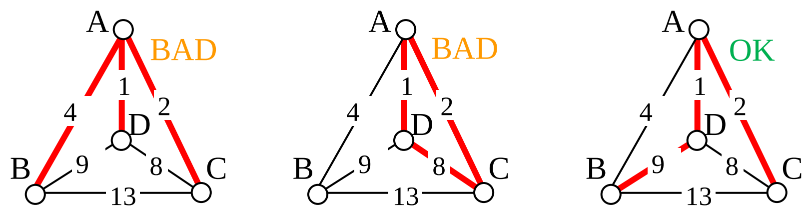
Using the four vertex graph from earlier, we can use the Cheapest Link algorithm.

The cheapest edge is AD, with a cost of 1. We highlight that edge to mark it selected. The next shortest edge is AC, with a weight of 2, so we highlight that edge.



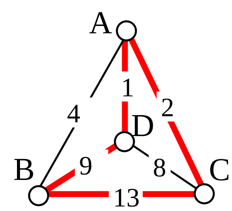


For the third edge, we'd like to add AB, but that would give vertex A degree 3, which is not allowed in a Hamiltonian circuit. The next shortest edge is CD, but that edge would create a circuit ACDA that does not include vertex B, so we reject that edge. The next shortest edge is BD, so we add that edge to the graph.



We then add the last edge to complete the circuit: ACBDA with weight 25.

Notice that the algorithm did not produce the optimal circuit in this case; the optimal circuit is ACDBA with weight 23.



While the Cheapest Link algorithm overcomes some of the shortcomings of NNA, it is still only a heuristic algorithm, and does not guarantee the optimal circuit.

Example 19

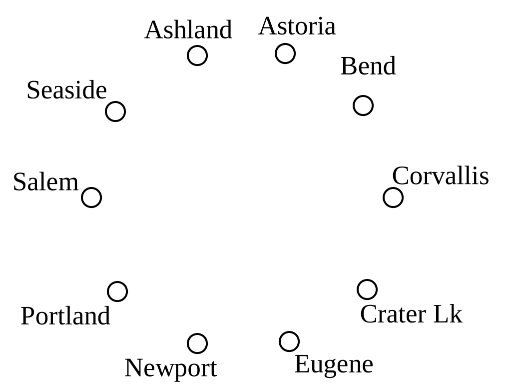
The author's band, *Derivative Work*, is doing a bar tour in Oregon. The driving distances are shown below. Plan an efficient route for your teacher to visit all the cities and return to the starting location. Use NNA starting at Portland, and then use Cheapest Link.

	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salem	Seaside
Ashland	-	374	200	223	108	178	252	285	240	356
Astoria	374	-	255	166	433	199	135	95	136	17
Bend	200	255	-	128	277	128	180	160	131	247
Corvallis	223	166	128	-	430	47	52	84	40	155
Crater Lake	108	433	277	430	-	453	478	344	389	423
Eugene	178	199	128	47	453	-	91	110	64	181
Newport	252	135	180	52	478	91	-	114	83	117
Portland	285	95	160	84	344	110	114	-	47	78
Salem	240	136	131	40	389	64	83	47	-	118
Seaside	356	17	247	155	423	181	117	78	118	-

Using NNA with a large number of cities, you might find it helpful to mark off the cities as they're visited to keep from accidently visiting them again. Looking in the row for Portland, the smallest distance is 47, to Salem. Continuing, our circuit will be:

Portland to Salem	47
Salem to Corvallis	40
Corvallis to Eugene	47
Eugene to Newport	91
Newport to Seaside	117
Seaside to Astoria	17
Astoria to Bend	255
Bend to Ashland	200
Ashland to Crater Lake	108
Crater Lake to Portland	344
Total trip length:	**1266 miles**

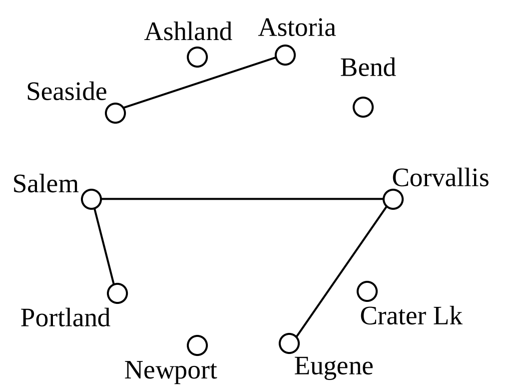
Using Cheapest Link, you might find it helpful to draw an empty graph, perhaps by drawing vertices in a circular pattern. Adding edges to the graph as you select them will help you visualize any circuits or vertices with degree 3.



We start adding the shortest edges:

Seaside to Astoria	17 miles
Corvallis to Salem	40 miles
Portland to Salem	47 miles
Corvallis to Eugene	47 miles

The graph after adding these edges is shown to the right.

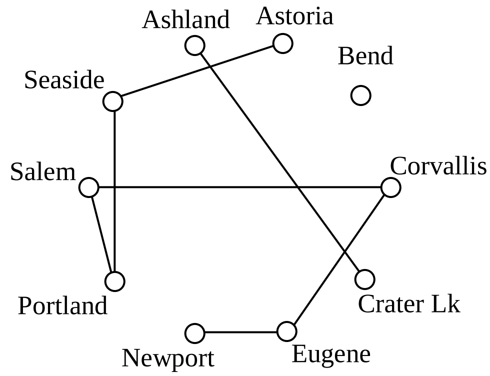


The next shortest edge is from Corvallis to Newport at 52 miles, but adding that edge would give Corvallis degree 3.

Continuing on, we can skip over any edge pair that contains Salem or Corvallis, since they both already have degree 2.

Portland to Seaside 78 miles
Eugene to Newport 91 miles
Portland to Astoria (reject – closes circuit)
Ashland to Crater Lk 108 miles

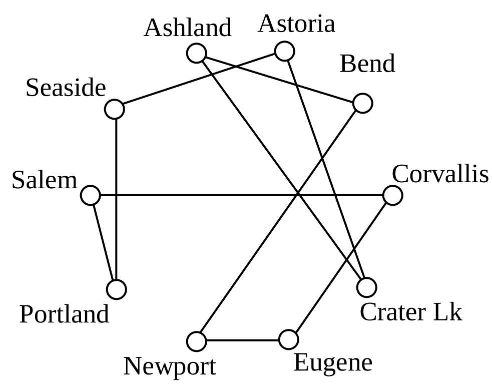
The graph after adding these edges is shown to the right.



At this point, we can skip over any edge pair that contains Salem, Seaside, Eugene, Portland, or Corvallis since they already have degree 2.

Newport to Astoria (reject – closes circuit)
Newport to Bend 180 miles
Bend to Ashland 200 miles

Now, the only way to complete the circuit is to add:
Crater Lk to Astoria 433 miles



The final circuit, written to start at Portland, is:

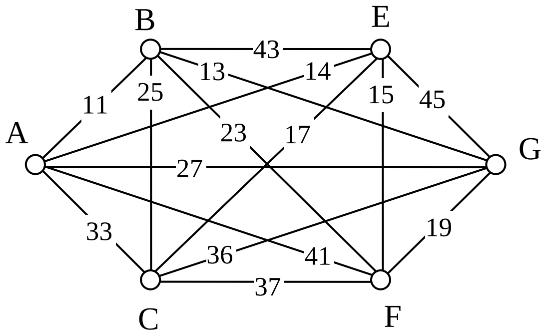
Portland, Salem, Corvallis, Eugene, Newport, Bend, Ashland, Crater Lake, Astoria, Seaside, Portland.

Total trip length: **1241 miles**.

While better than the NNA route, neither algorithm produced the optimal route. The following route can make the tour in 1069 miles:

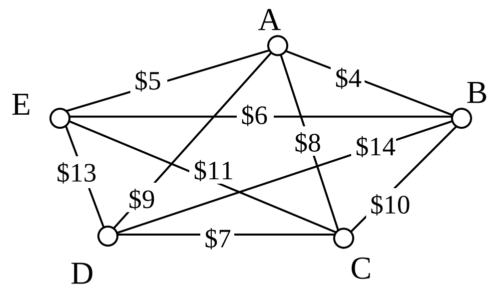
Portland, Astoria, Seaside, Newport, Corvallis, Eugene, Ashland, Crater Lake, Bend, Salem, Portland

Try it Now 5
Find the circuit produced by the Cheapest Link algorithm using the graph below.



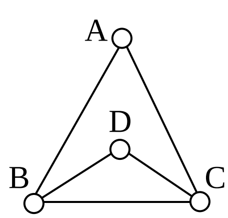
Sec. 1.4 Networks and Spanning Trees

A company requires reliable internet and phone connectivity between their five offices (named A, B, C, D, and E for simplicity) in New York, so they decide to lease dedicated lines from the phone company. The phone company will charge for each link made. The costs, in thousands of dollars per year, are shown in the graph.



Suppose that the company wants to connect their offices for the least possible annual cost. How might they go about this? First, it is clear that they do not need all of the edges; all we need to do is make sure we can make a call from any office to any other. In other words, we need to be sure there is a *path* from any vertex to any other vertex.

We don't need to find a circuit, or even a specific path. Instead we need a subset of the existing edges – that is, we want a **subgraph** of the original graph. Moreover, this graph should be connected and include all of the vertices. Finally, the optimal subgraph will not have any circuits. If this subgraph contains a circuit, then we can delete an edge without disconnecting the graph, and thus lower the cost. The reason is that if A and B are vertices in a circuit, then there are two paths from A to B. For example, in the following graph



ACDBA is a circuit. And we can see that there are two paths from A to D: ACD or ABD. So, even if we remove an edge from this circuit, the graph will still be connected.

A graph that is connected and has no circuits is called a **tree**. Trees show up in a variety of contexts, and play a major role in Graph Theory. So, we take a moment to list some important properties of trees.

Properties of Trees: Let G be a tree with n vertices. Then the following are true:

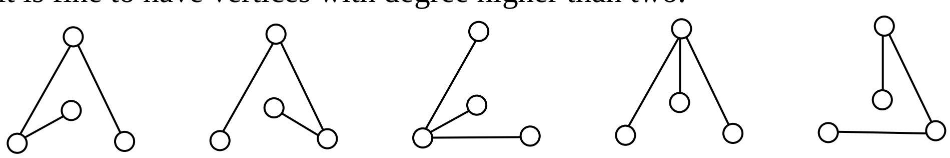
i) For any vertices u, v in G, there is a unique path from u to v.
ii) G has $n - 1$ edges.
iii) Every edge is a bridge; that is, if any edge of the graph is removed, then the remaining graph will be disconnected.

The third property tells us that a tree is a connected graph with the least number of edges. Of course, the subgraph we are after to create the optimal network is a special kind of tree:

Spanning Tree
A **spanning tree** for a graph G is a connected subgraph that uses all the vertices and in which there are no circuits.

In other words, there is a path from any vertex to any other vertex, but no circuits.

Some examples of spanning trees are shown below. Notice there are no circuits in the trees, and it is fine to have vertices with degree higher than two.



Usually we have a starting graph to work from, like in the phone example above. In this case, we form our spanning tree by finding a subgraph – a new graph formed using all the vertices but only some of the edges from the original graph. No edges will be created where they didn't already exist.

And of course, we don't want just any spanning tree. We want the **minimum cost spanning tree (MCST** or **MST)**.

Minimum Cost Spanning Tree (MCST)
The minimum cost spanning tree is the spanning tree with the smallest total edge weight.

In order to find a minimum cost spanning tree, we will take an approach similar to the Cheapest Link algorithm. That is, we'll systematically select edges, starting with the edge of least weight. At each step we will choose the lowest weight edge available, making sure that we do not complete a circuit. And we'll stop when all of the vertices are connected.

The algorithm we just described was discovered by the 20th century mathematician Joseph Kruskal:

Kruskal's Algorithm

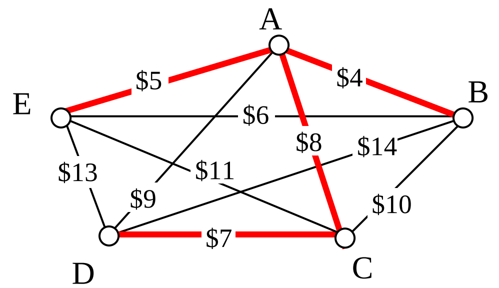
1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge, unless:
   a. adding the edge would create a circuit
3. Repeat until a spanning tree is formed

Remarkably, Kruskal's algorithm is both optimal and efficient; we are guaranteed to always produce the optimal MCST.

Example 20

Use Kruskal's Algorithm to find the minimum cost spanning tree for the phone line example at the start of the section. Using the graph from above, begin adding edges:

AB	\$4	OK
AE	\$5	OK
BE	\$6	reject – closes circuit ABEA
DC	\$7	OK
AC	\$8	OK



At this point we stop – every vertex is now connected, so we have formed a spanning tree with cost $24 thousand a year.

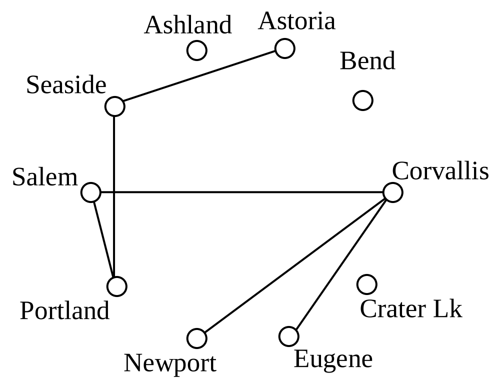
Example 21

The power company needs to lay updated distribution lines connecting the ten Oregon cities below to the power grid. How can they minimize the amount of new line to lay?

	Ashland	Astoria	Bend	Corvallis	Crater Lake	Eugene	Newport	Portland	Salem	Seaside
Ashland	-	374	200	223	108	178	252	285	240	356
Astoria	374	-	255	166	433	199	135	95	136	17
Bend	200	255	-	128	277	128	180	160	131	247
Corvallis	223	166	128	-	430	47	52	84	40	155
Crater Lake	108	433	277	430	-	453	478	344	389	423
Eugene	178	199	128	47	453	-	91	110	64	181
Newport	252	135	180	52	478	91	-	114	83	117
Portland	285	95	160	84	344	110	114	-	47	78
Salem	240	136	131	40	389	64	83	47	-	118
Seaside	356	17	247	155	423	181	117	78	118	-

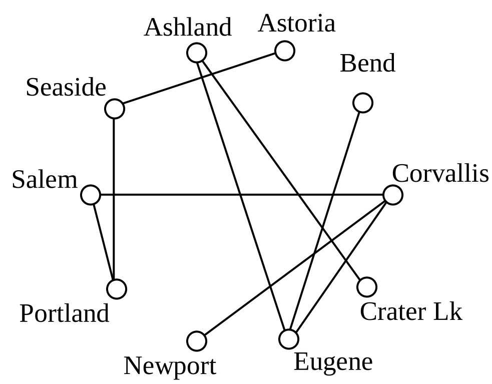
Using Kruskal's algorithm, we add edges from cheapest to most expensive, rejecting any that close a circuit. We stop when the graph is connected.

Seaside to Astoria	17 miles
Corvallis to Salem	40 miles
Portland to Salem	47 miles
Corvallis to Eugene	47 miles
Corvallis to Newport	52 miles
Salem to Eugene	reject – closes circuit
Portland to Seaside	78 miles



The graph up to this point is shown to the right. Continuing, we look at remaining edges:

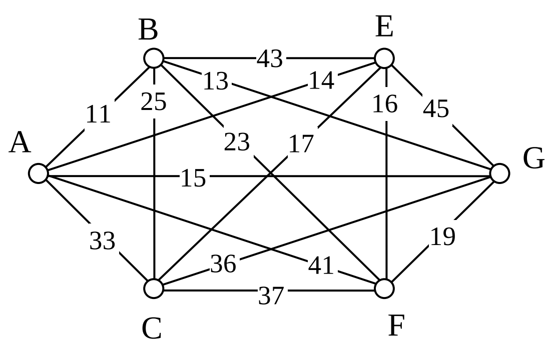
Newport to Salem	reject
Corvallis to Portland	reject
Eugene to Newport	reject
Portland to Astoria	reject
Ashland to Crater Lk	108 miles
Eugene to Portland	reject
Newport to Portland	reject
Newport to Seaside	reject
Salem to Seaside	reject
Bend to Eugene	128 miles
Bend to Salem	reject
Astoria to Newport	reject
Salem to Astoria	reject
Corvallis to Seaside	reject
Portland to Bend	reject
Astoria to Corvallis	reject
Eugene to Ashland	178 miles



This connects the graph. The total length of cable to lay would be **695 miles**.

Try it Now 6

Find a minimum cost spanning tree on the graph below using Kruskal's algorithm.



Sec. 1.5 Shortest Path Algorithm

When you visit a website like Google Maps or use your Smartphone to ask for directions from home to your Aunt's house in Pasadena, you are usually looking for a shortest path between the two locations. These computer applications use representations of the street maps as graphs, with estimated driving times as edge weights.

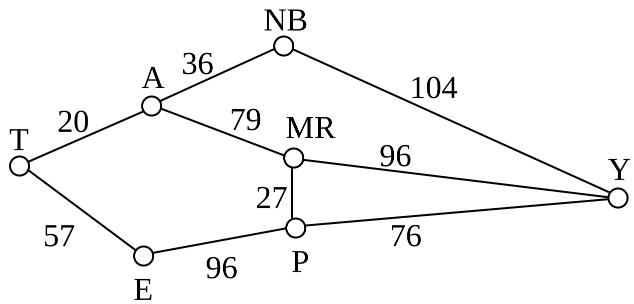
Moreover, they use an algorithm, called **Dijkstra's algorithm** ((pronounced dike-stra) to find the shortest path between two vertices.

Dijkstra's Algorithm

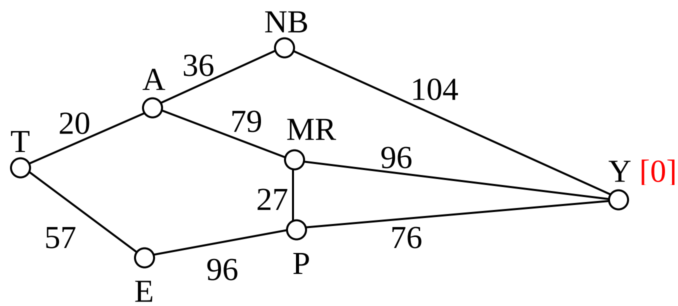
1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

Example 22

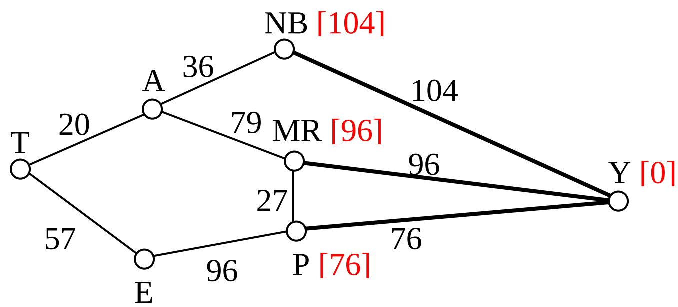
Suppose you need to travel from Tacoma, WA (vertex T) to Yakima, WA (vertex Y). Looking at a map, it looks like driving through Auburn (A) then Mount Rainier (MR) might be shortest, but it's not totally clear since that road is probably slower than taking the major highway through North Bend (NB). A graph with travel times in minutes is shown below. An alternate route through Eatonville (E) and Packwood (P) is also shown.



Step 1: Mark the ending vertex with a distance of zero. The distances will be recorded in [brackets] after the vertex name.



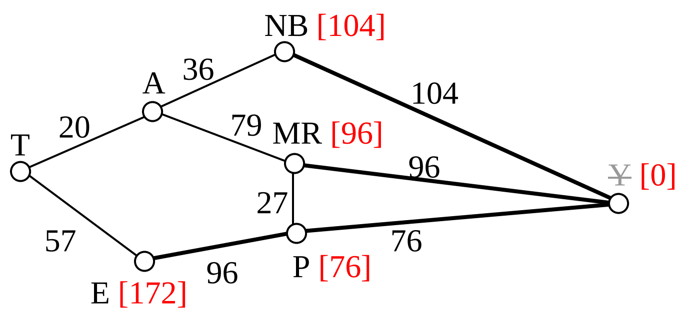
Step 2: For each vertex leading to Y, we calculate the distance to the end. For example, NB is a distance of 104 from the end, and MR is 96 from the end. Remember that distances in this case refer to the travel time in minutes.



Step 3 & 4: We mark Y as visited, and mark the vertex with the smallest recorded distance as current. At this point, P will be designated current. Back to step 2.

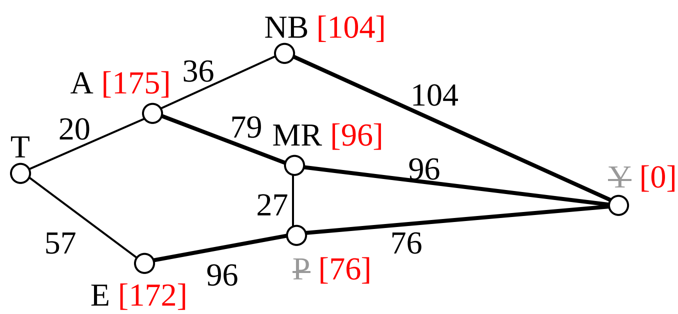
Step 2 (#2): For each vertex leading to P (and not leading to a visited vertex) we find the distance from the end. Since E is 96 minutes from P, and we've already calculated P is 76 minutes from Y, we can compute that E is 96+76 = 172 minutes from Y.

If we make the same computation for MR, we'd calculate 76+27 = 103. Since this is larger than the previously recorded distance from Y to MR, we will *not* replace it.



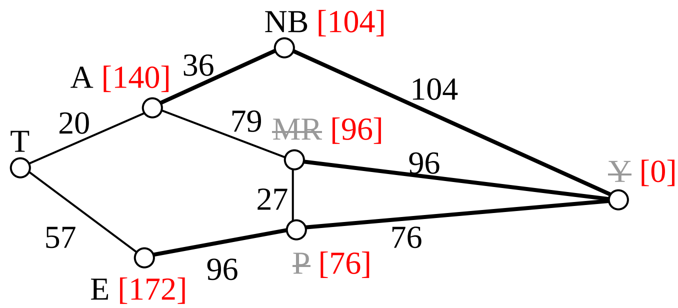
Step 3 & 4 (#2): We mark P as visited, and designate the vertex with the smallest recorded distance as current: MR. Back to step 2.

Step 2 (#3): For each vertex leading to MR (and not leading to a visited vertex) we find the distance to the end. The only vertex to be considered is A, since we've already visited Y and P. Adding MR's distance 96 to the length from A to MR gives the distance 96+79 = 175 minutes from A to Y.



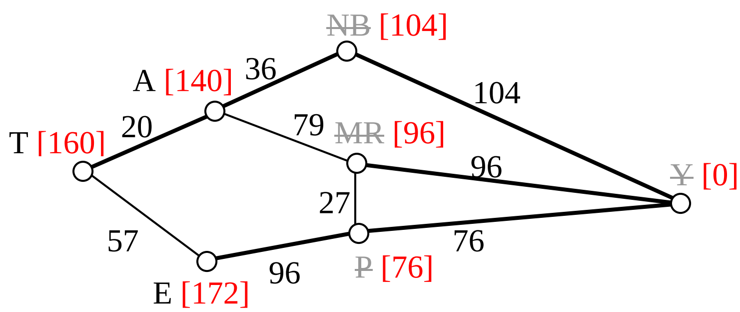
Step 3 & 4 (#3): We mark MR as visited, and designate the vertex with smallest recorded distance as current: NB. Back to step 2.

Step 2 (#4): For each vertex leading to NB, we find the distance to the end. We know the shortest distance from NB to Y is 104 and the distance from A to NB is 36, so the distance from A to Y through NB is 104+36 = 140. Since this distance *is* shorter than the previously calculated distance from Y to A through MR, we replace it.



Step 3 & 4 (#4): We mark NB as visited, and designate A as current, since it now has the shortest distance.

Step 2 (#5): T is the only non-visited vertex leading to A, so we calculate the distance from T to Y through A: 20+140 = 160 minutes.



Step 3 & 4 (#5): We mark A as visited, and designate E as current.

Step 2 (#6): The only non-visited vertex leading to E is T. Calculating the distance from T to Y through E, we compute 172+57 = 229 minutes. Since this is longer than the existing marked time, we do not replace it.

Step 3 (#6): We mark E as visited. Since all vertices have been visited, we are done.

From this, we know that the shortest path from Tacoma to Yakima will take 160 minutes. Tracking which sequence of edges yielded 160 minutes, we see the shortest path is T-A-NB-Y.

Dijkstra's algorithm is an **optimal algorithm**, meaning that it always produces the actual shortest path, not just a path that is pretty short, provided one exists. This algorithm is also **efficient**, meaning that it can be implemented in a reasonable amount of time. Dijkstra's algorithm takes around V^2 calculations, where V is the number of vertices in a graph[4]. A graph with 100 vertices would take around 10,000 calculations. While that would be a lot to do by hand, it is not a lot for computer to handle. It is because of this efficiency that your car's GPS unit can compute driving directions in only a few seconds.

[4] It can be made to run faster through various optimizations to the implementation.

In contrast, an **inefficient** algorithm might try to list all possible paths then compute the length of each path. Trying to list all possible paths could easily take 10^{25} calculations to compute the shortest path with only 25 vertices; that's a 1 with 25 zeros after it! To put that in perspective, the fastest computer in the world would still spend over 1000 years analyzing all those paths.

Example 23

A shipping company needs to route a package from Washington, D.C. to San Diego, CA. To minimize costs, the package will first be sent to their processing center in Baltimore, MD then sent as part of mass shipments between their various processing centers, ending up in their processing center in Bakersfield, CA. From there it will be delivered in a small truck to San Diego.

The travel times, in hours, between their processing centers are shown in the table below. Three hours has been added to each travel time for processing. Find the shortest path from Baltimore to Bakersfield.

	Baltimore	Denver	Dallas	Chicago	Atlanta	Bakersfield
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

While we could draw a graph, we can also work directly from the table.

Step 1: The ending vertex, Bakersfield, is marked as current.

Step 2: All cities connected to Bakersfield, in this case Denver and Dallas, have their distances calculated; we'll mark those distances in the column headers.

Step 3 & 4: Mark Bakersfield as visited. Here, we are doing it by shading the corresponding row and column of the table. We mark Denver as current, shown in bold, since it is the vertex with the shortest distance.

	Baltimore	**Denver** [19]	Dallas [25]	Chicago	Atlanta	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#2): For cities connected to Denver, calculate distance to the end. For example, Chicago is 18 hours from Denver, and Denver is 19 hours from the end, the distance for Chicago to the end is 18+19 = 37 (Chicago to Denver to Bakersfield). Atlanta is 24 hours from Denver, so the distance to the end is 24+19 = 43 (Atlanta to Denver to Bakersfield).

Step 3 & 4 (#2): We mark Denver as visited and mark Dallas as current.

	Baltimore	Denver [19]	**Dallas** [25]	Chicago [37]	Atlanta [43]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#3): For cities connected to Dallas, calculate the distance to the end. For Chicago, the distance from Chicago to Dallas is 18 and from Dallas to the end is 25, so the distance from Chicago to the end through Dallas would be 18+25 = 43. Since this is longer than the currently marked distance for Chicago, we do not replace it. For Atlanta, we calculate 15+25 = 40. Since this is shorter than the currently marked distance for Atlanta, we replace the existing distance.

Step 3 & 4 (#3): We mark Dallas as visited, and mark Chicago as current.

	Baltimore	Denver [19]	Dallas [25]	**Chicago** [37]	Atlanta [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

Step 2 (#4): Baltimore and Atlanta are the only non-visited cities connected to Chicago. For Baltimore, we calculate 15+37 = 52 and mark that distance. For Atlanta, we calculate 14+37 = 51. Since this is longer than the existing distance of 40 for Atlanta, we do not replace that distance.

Step 3 & 4 (#4): Mark Chicago as visited and Atlanta as current.

	Baltimore [52]	Denver [19]	Dallas [25]	Chicago [37]	**Atlanta** [40]	Bakersfield [0]
Baltimore	*			15	14	
Denver		*		18	24	19
Dallas			*	18	15	25
Chicago	15	18	18	*	14	
Atlanta	14	24	15	14	*	
Bakersfield		19	25			*

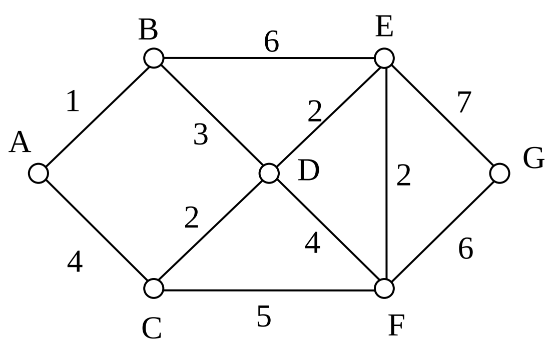
Step 2 (#5): The distance from Atlanta to Baltimore is 14. Adding that to the distance already calculated for Atlanta gives a total distance of 14+40 = 54 hours from Baltimore to Bakersfield through Atlanta. Since this is larger than the currently calculated distance, we do not replace the distance for Baltimore.

Step 3 & 4 (#5): We mark Atlanta as visited. All cities have been visited and we are done.

The shortest route from Baltimore to Bakersfield will take 52 hours, and will route through Chicago and Denver.

Try it Now 7

Find the shortest path between vertices A and G in the graph below.

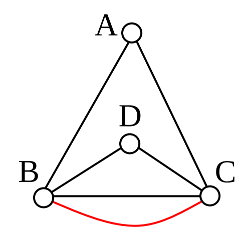


Try it Now Answers

1. a. 5 vertices, 10 edges
b. Yes, it is connected
c. The vertex is degree 4
d. A path
e. A circuit

2. Yes, all vertices have even degree so this graph has an Euler Circuit. There are several possibilities. One is: ABEGFCDFEDBCA

3. This graph can be eulerized by duplicating the edge BC, as shown. One possible Euler circuit on the eulerized graph is ACDBCBA



4. At each step, we look for the nearest location we haven't already visited.
From B the nearest computer is E with time 24.
From E, the nearest computer is D with time 11.
From D the nearest is A with time 12.
From A the nearest is C with time 34.
From C, the only computer we haven't visited is F with time 27
From F, we return back to B with time 50.

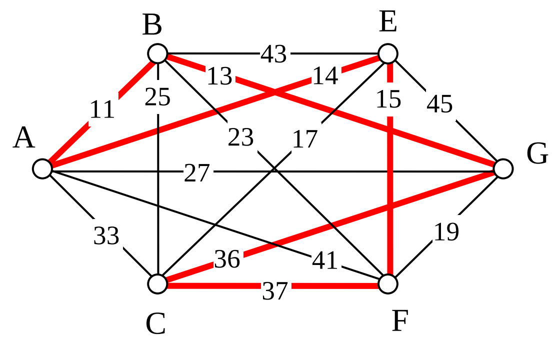
The NNA circuit from B is BEDACFB with time 158 milliseconds.

Using NNA again from other starting vertices:
Starting at A: ADEBCFA: time 146
Starting at C: CDEBAFC: time 167
Starting at D: DEBCFAD: time 146
Starting at E: EDACFBE: time 158
Starting at F: FDEBCAF: time 158

The RNNA found a circuit with time 146 milliseconds: ADEBCFA. We could also write this same circuit starting at B if we wanted: BCFADEB or BEDAFCB.

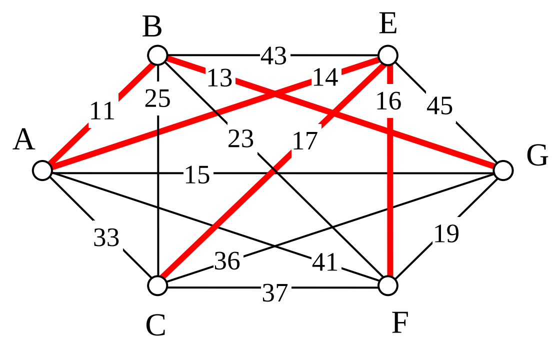
Try it Now Answers Continued

5.
AB: Add, cost 11
BG: Add, cost 13
AE: Add, cost 14
EF: Add, cost 15
EC: Skip (degree 3 at E)
FG: Skip (would create a circuit not including C)
BF, BC, AG, AC: Skip (would cause a vertex to have degree 3)
GC: Add, cost 36
CF: Add, cost 37, completes the circuit



Final circuit: ABGCFEA

6.
AB: Add, cost 11
BG: Add, cost 13
AE: Add, cost 14
AG: Skip, would create circuit ABGA
EF: Add, cost 16
EC: Add, cost 17



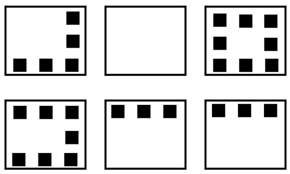
This completes the spanning tree

7. The shortest path is ABDEG, with length 13

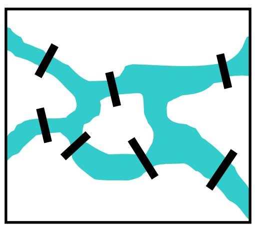
Exercises

Skills

1. To deliver mail in a particular neighborhood, the postal carrier needs to walk along each of the streets with houses (the dots). Create a graph with edges showing where the carrier must walk to deliver the mail.



2. Suppose that a town has 7 bridges as pictured below. Create a graph that could be used to determine if there is a path that crosses all bridges once.



3. The table below shows approximate driving times (in minutes, without traffic) between five cities in the Dallas area. Create a weighted graph representing this data.

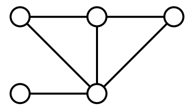
	Fort Worth	Plano	Mesquite	Arlington	Denton
Fort Worth	-	54	52	19	42
Plano	54	-	38	53	41
Mesquite	52	38	-	43	56
Arlington	19	53	43	-	50
Denton	42	41	56	50	-

4. Shown in the table below are the one-way airfares between 5 cities[5]. Create a graph showing this data.

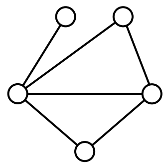
	Seattle	Honolulu	London	Moscow	Cairo
Seattle	-	$159	$370	$654	$684
Honolulu	$159	-	$830	$854	$801
London	$370	$830	-	$245	$323
Moscow	$654	$854	$245	-	$329
Cairo	$684	$323	$323	$329	-

[5] Cheapest fares found when retrieved Sept 1, 2009 for travel Sept 22, 2009

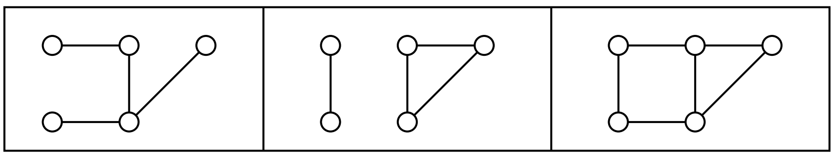
5. Find the degree of each vertex in the graph below.



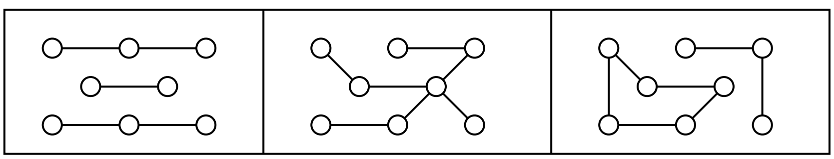
6. Find the degree of each vertex in the graph below.



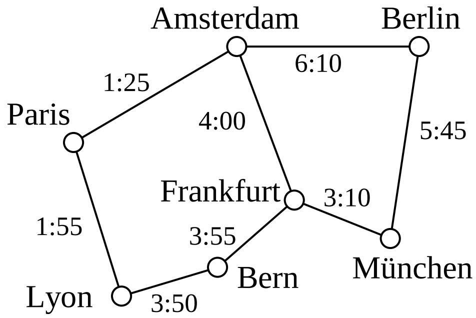
7. Which of these graphs are connected?



8. Which of these graphs are connected?



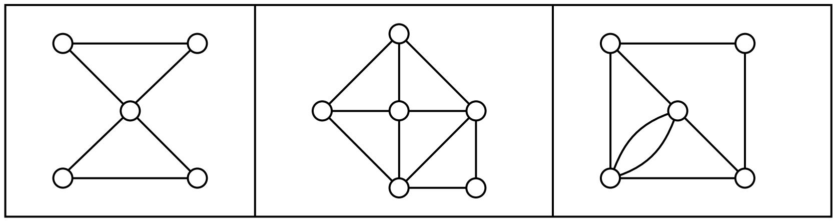
9. Travel times by rail for a segment of the Eurail system is shown below with travel times in hours and minutes[6]. Find path with shortest travel time from Bern to Berlin by applying Dijkstra's algorithm.



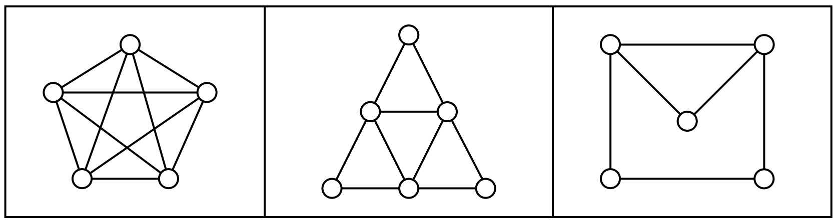
10. Using the graph from the previous problem, find the path with shortest travel time from Paris to München.

[6] From http://www.eurail.com/eurail-railway-map

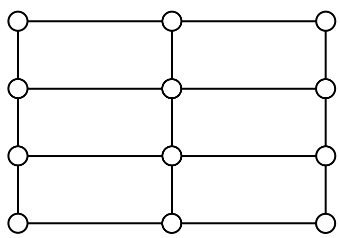
11. Does each of these graphs have an Euler circuit? If so, find it.



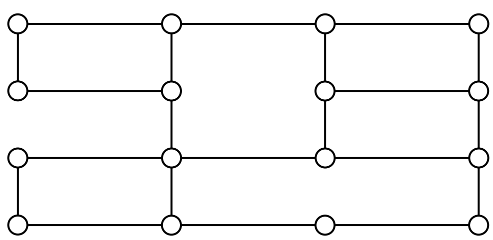
12. Does each of these graphs have an Euler circuit? If so, find it.



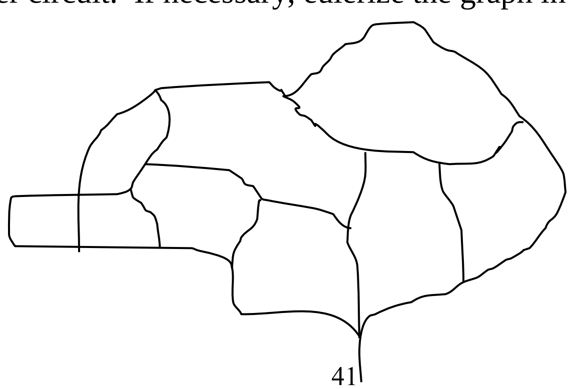
13. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



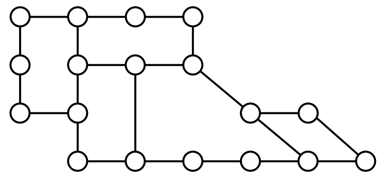
14. Eulerize this graph using as few edge duplications as possible. Then, find an Euler circuit.



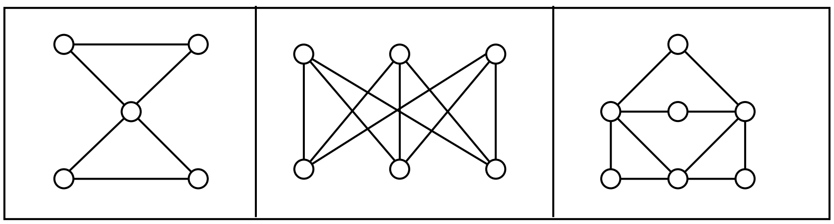
15. The maintenance staff at an amusement park need to patrol the major walkways, shown in the graph below, collecting litter. Find an efficient patrol route by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



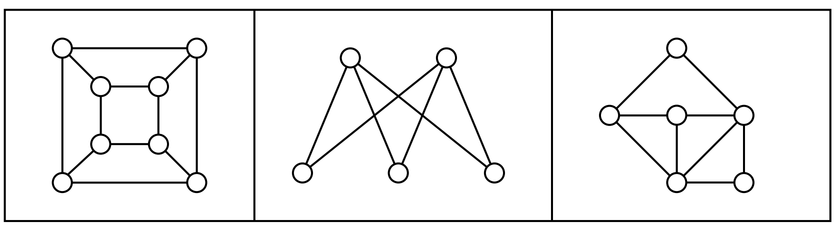
16. After a storm, the city crew inspects for trees or brush blocking the road. Find an efficient route for the neighborhood below by finding an Euler circuit. If necessary, eulerize the graph in an efficient way.



17. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



18. Does each of these graphs have at least one Hamiltonian circuit? If so, find one.



19. A company needs to deliver product to each of their 5 stores around the Dallas, TX area. Driving distances between the stores are shown below. Find a route for the driver to follow, returning to the distribution center in Fort Worth:

   a. Using Nearest Neighbor starting in Fort Worth
   b. Using Repeated Nearest Neighbor
   c. Using Cheapest Link

	Fort Worth	Plano	Mesquite	Arlington	Denton
Fort Worth	-	54	52	19	42
Plano	54	-	38	53	41
Mesquite	52	38	-	43	56
Arlington	19	53	43	-	50
Denton	42	41	56	50	-

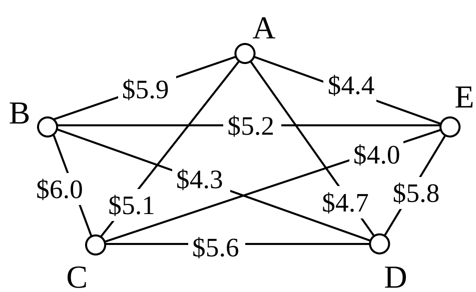
20. A salesperson needs to travel from Seattle to Honolulu, London, Moscow, and Cairo. Use the table of flight costs from problem #4 to find a route for this person to follow:

   a. Using Nearest Neighbor starting in Seattle
   b. Using Repeated Nearest Neighbor
   c. Using Cheapest Link

	Seattle	Honolulu	London	Moscow	Cairo
Seattle	-	$159	$370	$654	$684
Honolulu	$159	-	$830	$854	$801
London	$370	$830	-	$245	$323
Moscow	$654	$854	$245	-	$329
Cairo	$684	$323	$323	$329	-

21. When installing fiber optics, some companies will install a sonet ring; a full loop of cable connecting multiple locations. This is used so that if any part of the cable is damaged it does not interrupt service, since there is a second connection to the hub. A company has 5 buildings. Costs (in thousands of dollars) to lay cables between pairs of buildings are shown below. Find the circuit that will minimize cost:

   a. Using Nearest Neighbor starting at building A
   b. Using Repeated Nearest Neighbor
   c. Using Cheapest Link



22. A tourist wants to visit 7 cities in Israel. Driving distances, in kilometers, between the cities are shown below[7]. Find a route for the person to follow, returning to the starting city:
    a. Using Nearest Neighbor starting in Jerusalem
    b. Using Repeated Nearest Neighbor
    c. Using Cheapest Link

	Jerusalem	Tel Aviv	Haifa	Tiberias	Beer Sheba	Eilat	Nazareth
Jerusalem	--	58	151	152	81	309	131
Tel Aviv	58	--	95	134	105	346	102
Haifa	151	95	--	69	197	438	35
Tiberias	152	134	69	--	233	405	29
Beer Sheba	81	105	197	233	--	241	207
Eilat	309	346	438	405	241	--	488
Nazareth	131	102	35	29	207	488	--

23. Find a minimum cost spanning tree for the graph you created in problem #3

24. Find a minimum cost spanning tree for the graph you created in problem #22

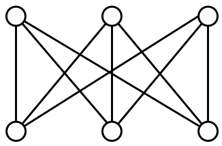
25. Find a minimum cost spanning tree for the graph from problem #21

Concepts

26. Can a graph have one vertex with odd degree? If not, are there other values that are not possible? Why?

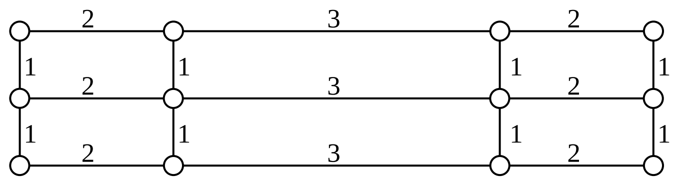
27. A complete graph is one in which there is an edge connecting every vertex to every other vertex. For what values of n does complete graph with n vertices have an Euler circuit? A Hamiltonian circuit?

28. Create a graph by drawing n vertices in a row, then another n vertices below those. Draw an edge from each vertex in the top row to every vertex in the bottom row. An example when n=3 is shown below. For what values of n will a graph created this way have an Euler circuit? A Hamiltonian circuit?

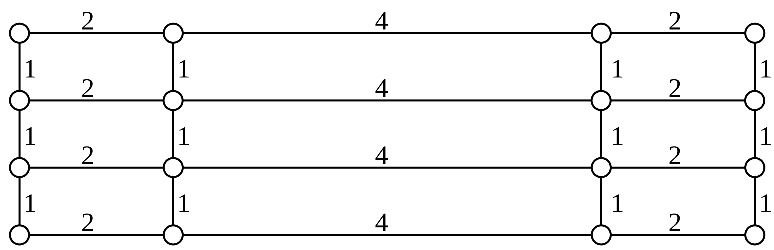


[7] From http://www.ddtravel-acc.com/Israel-cities-distance.htm

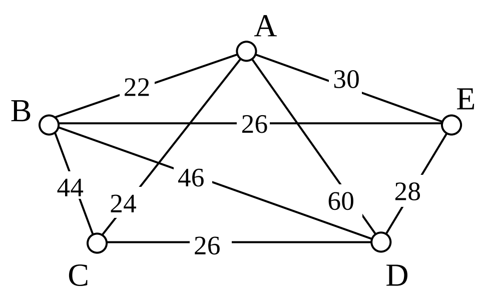
29. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



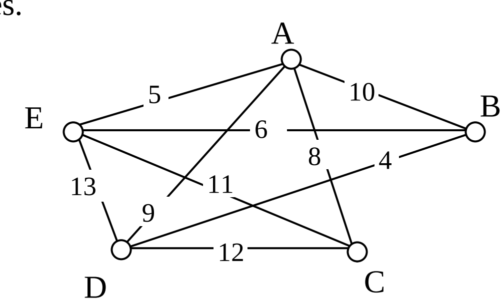
30. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



31. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



32. Eulerize this graph in the most efficient way possible, considering the weights of the edges.



Explorations

33. Social networks such as Facebook and LinkedIn can be represented using graphs in which vertices represent people and edges are drawn between two vertices when those people are "friends." The table below shows a friendship table, where an X shows that two people are friends.

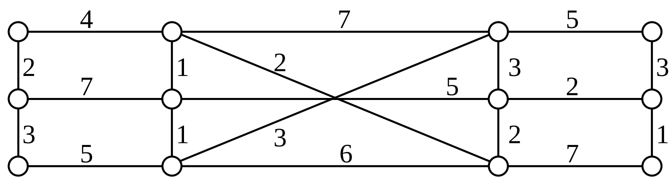
	A	B	C	D	E	F	G	H	I
A		X	X			X	X		
B			X		X				
C					X				
D					X				X
E							X		X
F								X	X
G								X	
H									X

   a. Create a graph of this friendship table
   b. Find the shortest path from A to D. The length of this path is often called the "degrees of separation" of the two people.
   c. Extension: Split into groups. Each group will pick 10 or more movies, and look up their major actors (www.imdb.com is a good source). Create a graph with each actor as a vertex, and edges connecting two actors in the same movie (note the movie name on the edge). Find interesting paths between actors, and quiz the other groups to see if they can guess the connections.

34. A spell checker in a word processing program makes suggestions when it finds a word not in the dictionary. To determine what words to suggest, it tries to find similar words. One measure of word similarity is the Levenshtein distance, which measures the number of substitutions, additions, or deletions that are required to change one word into another. For example, the words spit and spot are a distance of 1 apart; changing spit to spot requires one substitution (i for o). Likewise, spit is distance 1 from pit since the change requires one deletion (the s). The word spite is also distance 1 from spit since it requires one addition (the e). The word soot is distance 2 from spit since two substitutions would be required.
   a. Create a graph using words as vertices, and edges connecting words with a Levenshtein distance of 1. Use the misspelled word "moke" as the center, and try to find at least 10 connected dictionary words. How might a spell checker use this graph?
   b. Improve the method from above by assigning a weight to each edge based on the likelihood of making the substitution, addition, or deletion. You can base the weights on any reasonable approach: proximity of keys on a keyboard, common language errors, etc. Use Dijkstra's algorithm to find the length of the shortest path from each word to "moke". How might a spell checker use these values?

35. The graph below contains two vertices of odd degree. To eulerize this graph, it is necessary to duplicate edges connecting those two vertices.

  a. Use Dijkstra's algorithm to find the shortest path between the two vertices with odd degree. Does this produce the most efficient eulerization and solve the Chinese Postman Problem for this graph?



  b. Suppose a graph has *n* odd vertices. Using the approach from part a, how many shortest paths would need to be considered? Is this approach going to be efficient?

Chapter 2
Scheduling

Excerpted from *Math in Society*, by David Lippman

Scheduling

An event planner has to juggle many workers completing different tasks, some of which must be completed before others can begin. For example, the banquet tables would need to be arranged in the room before the catering staff could begin setting out silverware. The event planner has to carefully schedule things so that everything gets done in a reasonable amount of time.

The problem of scheduling is fairly universal. Contractors need to schedule workers and subcontractors to build a house as quickly as possible. A magazine needs to schedule writers, editors, photographers, typesetters, and others so that an issue can be completed on time.

Section 2.1 Getting Started

To begin thinking about scheduling, let us consider an auto shop that is converting a car from gas to electric. A number of steps are involved. A time estimate for each task is given.

Task 1: Remove engine and gas parts (2 days)
Task 2: Steam clean the inside of the car (0.5 day)
Task 3: Buy an electric motor and speed controller (2 days for travel)
Task 4: Construct the part that connects the motor to the car's transmission (1 day)
Task 5: Construct battery racks (2 days)
Task 6: Install the motor (0.5 day)
Task 7: Install the speed controller (0.5 day)
Task 8: Install the battery racks (0.5 day)
Task 9: Wire the electricity (1 day)

Some tasks have to be completed before others – we certainly can't install the new motor before removing the old engine! There are some tasks, however, that can be worked on simultaneously by two different people, like constructing the battery racks and installing the motor.

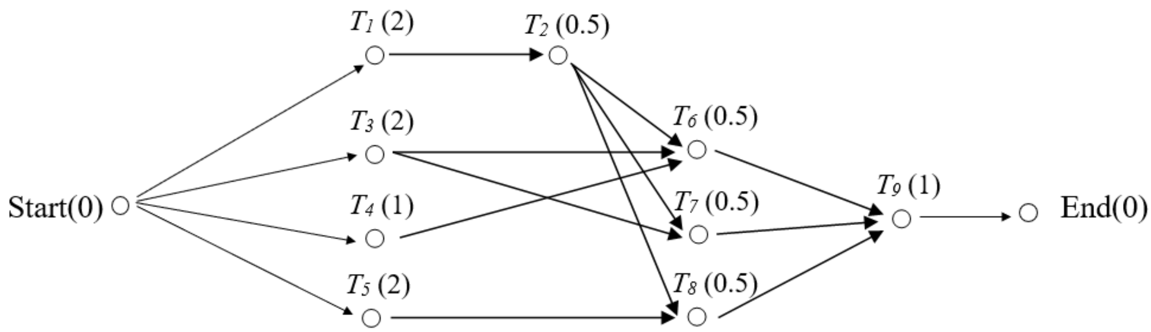
To help us visualize the ordering of tasks, we will create a **digraph.**

Digraph
A diagraph is a graphical representation of a set of tasks in which tasks are represented with dots or circles, called vertices, and arrows between vertices are used to show ordering.

For example, this digraph shows that Task 1, notated T_1 for compactness, needs to be completed before Task 2. The number in parentheses after the task name is the time required for the task.

Example 1

A complete digraph for our car conversion would look like this:



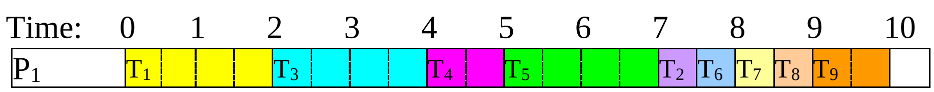
The time it takes to complete this job will partially depend upon how many people are working on the project.

> **Processors**
> In scheduling jargon, the workers completing the tasks are called **processors**. While in this example the processors are humans, in some situations the processors are computers, robots, or other machines.

For simplicity, we are going to make the very big assumptions that every processor can do every task, that they all would take the same time to complete it, and that only one processor can work on a task at a time.

> **Finishing Time**
> The **finishing** time is how long it will take to complete all the tasks. The finishing time will depend upon the number of processors and the specific schedule.

If we had only one processor working on this task, it is easy to determine the finishing time; just add up the individual times. We assume one person can't work on two tasks at the same time, ignore things like drying times during which someone could work on another task. Scheduling with one processor, a possible schedule would look like this, with a finishing time of 10 days.

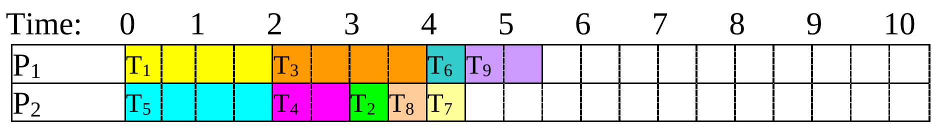


In this schedule, all the ordering requirements are met. This is certainly not the only possible schedule for one processor, but no other schedule could complete the job in less time.

Because of this, this is an **optimal schedule** with **optimal finishing time** – there is nothing better.

> **Optimal Schedule**
> An optimal schedule is the schedule with the shortest possible finishing time.

For two processors, things become more interesting. For small digraphs like this, we probably could fiddle around and guess-and-check a pretty good schedule. Here would be a possibility:

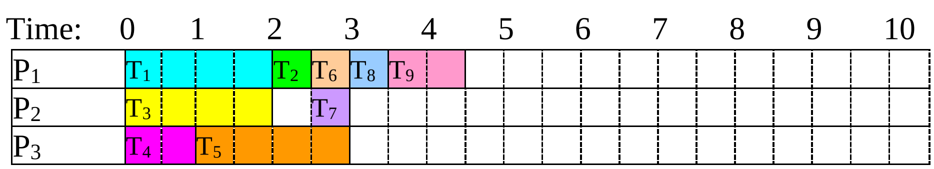


With two processors, the finishing time was reduced to 5.5 days. What was processor 2 doing during the last day? Nothing, because there were no tasks for the processor to do. This is called **idle time**.

> **Idle Time**
> Idle time is time in the schedule when there are no tasks available for the processor to work on, so they sit idle.

Is this schedule optimal? Could it have been completed in 5 days? Because every other task had to be completed before task 9 could start, there would be no way that both processors could be busy during task 9, so it is not possible to create a shorter schedule.

So how long will it take if we use three processors? About 10/3 = 3.33 days? Again we will guess-and-check a schedule:



With three processors, the job still took 4.5 days. It is a little harder to tell whether this schedule is optimal. However, it might be helpful to notice that since Task 1, 2, 6, and 9 have to be completed sequentially, there is no way that this job could be completed in less than 2+0.5+0.5+1 = 4 days, regardless of the number of processors. Four days is, for this digraph, the absolute minimum time to complete the job, called the **critical time**.

> **Critical Time**
> The critical time is the absolute minimum time to complete the job, regardless of the number of processors working on the tasks.
>
> Critical time can be determined by looking at the longest sequence of tasks in the digraph, called the critical path

Sec. 2.2 List Processing Algorithm

Until now, we have been creating schedules by guess-and-check, which works well enough for small schedules, but would not work well with dozens or hundreds of tasks. To create a more procedural approach, we might begin by somehow creating a **priority list**.

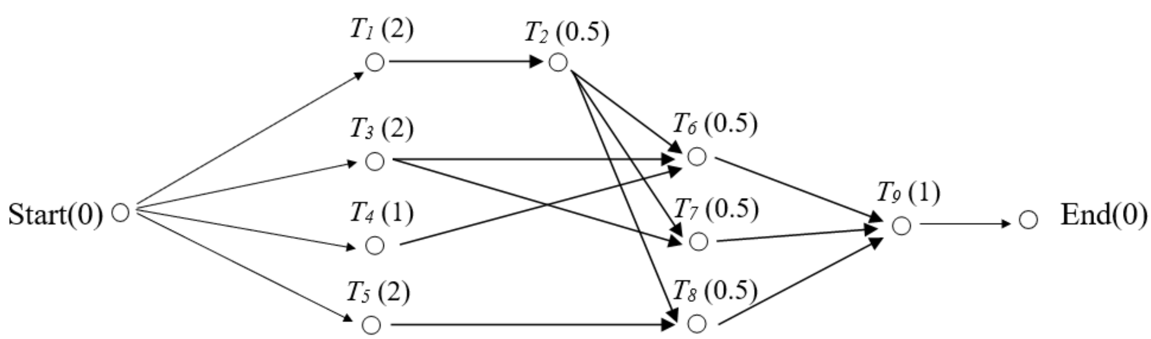
> **Priority List**
> A priority list is a list of tasks given in the order in which we desire them to be completed.

The priority list may be imposed on us due to clients' needs, or the priorities of our company. Once we have a priority list, we can begin scheduling using the **list processing algorithm**:

> **List Processing Algorithm**
> 1. On the digraph or priority list, circle all tasks that are ready, meaning that all pre-requisite tasks have been completed.
> 2. Assign to each available processor, in order, the first ready task. Mark the task as in progress, perhaps by putting a single line through the task.
> 3. Move forward in time until a task is completed. Mark the task as completed, perhaps by crossing out the task. If any new tasks become ready, mark them as such.
> 4. Repeat until all tasks have been scheduled.

Example 2

Using the digraph from Example 1,



Schedule the project with two processors, using the priority list:

T_1, T_3, T_4, T_5, T_6, T_7, T_8, T_2, T_9

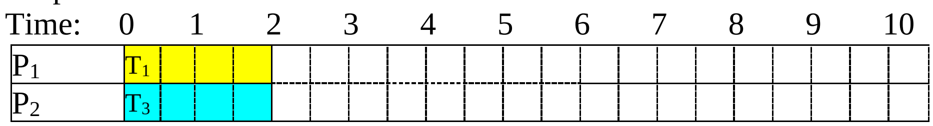
Time 0: Mark ready tasks

Priority list: T_1, T_3, T_4, T_5, T_6, T_7, T_8, T_2, T_9

We assign the first task, T_1 to the first processor, P_1, and the second ready task, T_3, to the second processor. Making those assignments, we mark those tasks as in progress:

Priority list: ~~T_1,~~ ~~T_3,~~ T_4, T_5, T_6, T_7, T_8, T_2, T_9

Schedule up to here:



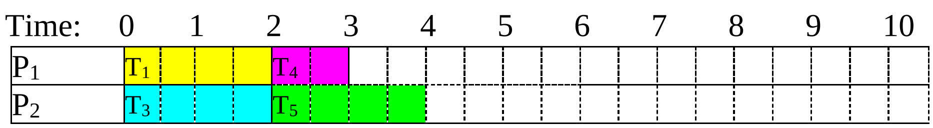
We jump to the time when the next task completes, which is at time 2.

Time 2: Both processors complete their tasks. We mark those tasks as complete. With Task 1 complete, Task 2 becomes ready:

Priority list: ~~T_1,~~ ~~T_3,~~ T_4, T_5, T_6, T_7, T_8, T_2, T_9

We assign the next ready task on the list, T_4 to P_1, and T_5 to P_2.

Priority list: ~~T_1,~~ ~~T_3,~~ ~~T_4,~~ ~~T_5,~~ T_6, T_7, T_8, T_2, T_9

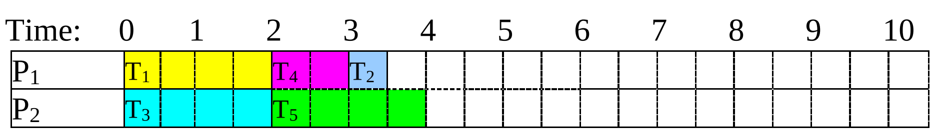


Time 3: Processor 1 has completed T_4. Completing T_4 does not make any other tasks ready (note that all the rest require that T_2 be completed first).

Priority list: ~~T_1,~~ ~~T_3,~~ ~~T_4,~~ ~~T_5,~~ T_6, T_7, T_8, T_2, T_9

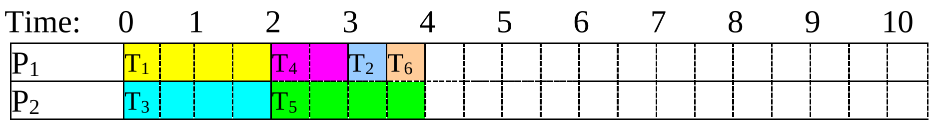
Since the next three tasks are not yet ready, we assign the next ready task, T_2 to P_1

Priority list: ~~T_1,~~ ~~T_3,~~ ~~T_4,~~ ~~T_5,~~ T_6, T_7, T_8, ~~T_2,~~ T_9



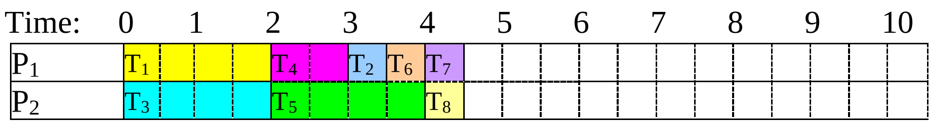
Time 3.5: Processor 1 has completed T_2. Completing T_2 causes T_6 and T_7 to become ready. We assign T_6 to P_1

Priority list: ~~T_1,~~ ~~T_3,~~ ~~T_4,~~ ~~T_5,~~ ~~T_6,~~ T_7, T_8, ~~T_2,~~ T_9



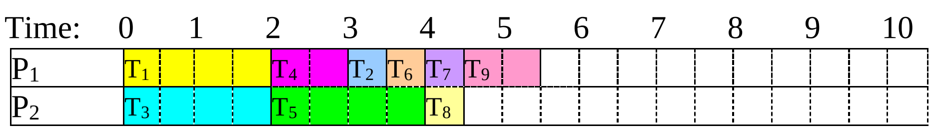
Time 4: Both processors complete their tasks. The completion of T_5 allows T_8 to become ready. We assign T_7 to P_1 and T_8 to P_2.

Priority list: T_1, T_3, T_4, T_5, T_6, T_7, ~~T_8~~, T_2, T_9



Time 4.5: Both processors complete their tasks. T_9 becomes ready, and is assigned to P_1. There is no ready task for P_2 to work on, so P_2 idles.

Priority list: ~~T_1~~, ~~T_3~~, ~~T_4~~, ~~T_5~~, ~~T_6~~, ~~T_7~~, ~~T_8~~, ~~T_2~~, ~~T_9~~

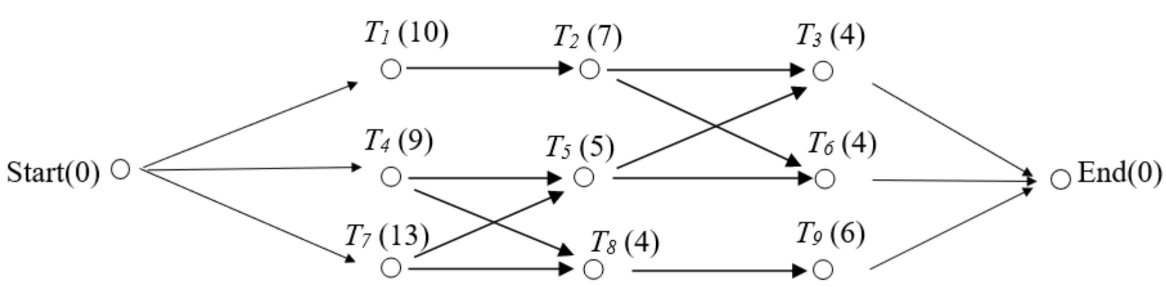


With the last task completed, we have a completed schedule, with finishing time 5.5 days.

Try it Now 1

Using the digraph below, create a schedule using two processors and the priority list:

T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9



It is important to note that the list processing algorithm itself does not influence the resulting schedule – the schedule is completely determined by number of processors and the priority list followed. The list processing, while do-able by hand, could just as easily be executed by a computer. The interesting part of scheduling, then, is how to choose a priority list that will create the best possible schedule. Next, we will explore two algorithms for selecting a priority list.

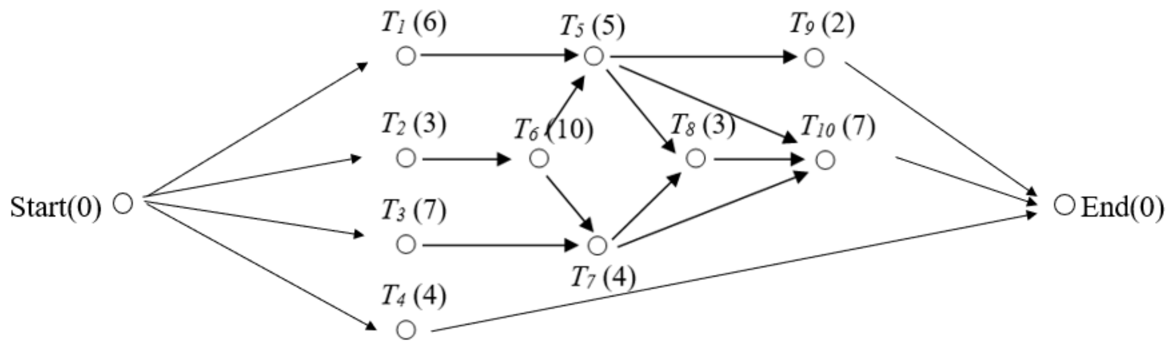
Sec. 2.3 Decreasing time algorithm

The decreasing time algorithm takes the approach of trying to get the very long tasks out of the way as soon as possible by putting them first on the priority list.

> **Decreasing Time Algorithm**
> Create the priority list by listing the tasks in order from longest completion time to shortest completion time.

Example 3

Consider the scheduling problem represented by the digraph below. Create a priority list using the decreasing time list algorithm, then use it to schedule for two processors using the list processing algorithm.



To use the decreasing time list algorithm, we create our priority list by listing the tasks in order from longest task time to shortest task time. If there is a tie, we will list the task with smaller task number first (not for any good reason, but just for consistency).

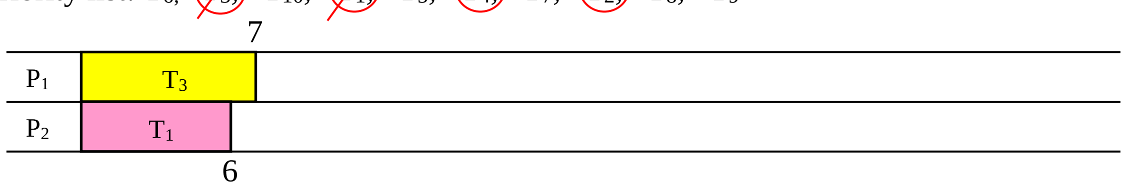
For this digraph, the decreasing time algorithm would create a priority list of:

T_6 (10), T_3 (7), T_{10} (7), T_1 (6), T_5 (5), T_4 (4), T_7 (4), T_2 (3), T_8 (3), T_9 (2)

Once we have the priority list, we can create the schedule using the list processing algorithm. With two processors, we'd get:

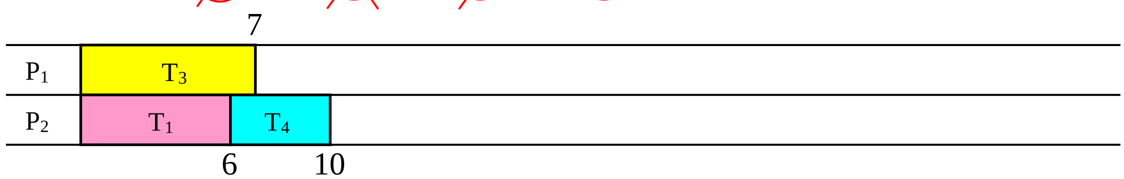
Time 0: We identify ready tasks, and assign T_3 to P_1 and T_1 to P_2

Priority list: T_6, $\cancel{T_3}$, T_{10}, $\cancel{T_1}$, T_5, T_4, T_7, T_2, T_8, T_9



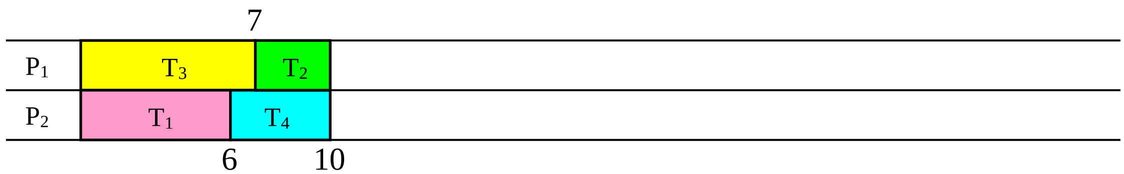
Time 6: P_2 completes T_1. No new tasks become ready, so T_4 is assigned to P_2.

Priority list: T_6, ~~T_3,~~ T_{10}, ~~T_1,~~ T_5, ~~T_4,~~ T_7, T_2, T_8, T_9



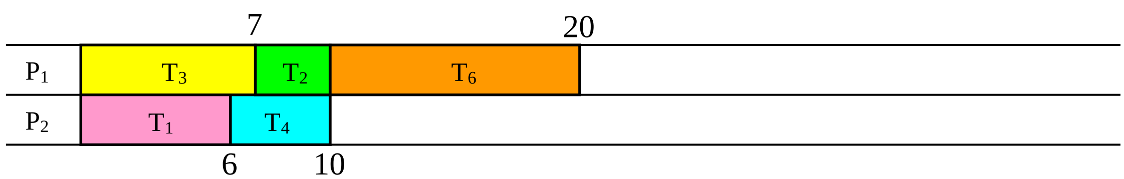
Time 7: P_1 completes T_3. No new tasks become ready, so T_2 is assigned to P_1.

Priority list: T_6, ~~T_3,~~ T_{10}, ~~T_1,~~ T_5, ~~T_4,~~ T_7, ~~T_2,~~ T_8, T_9



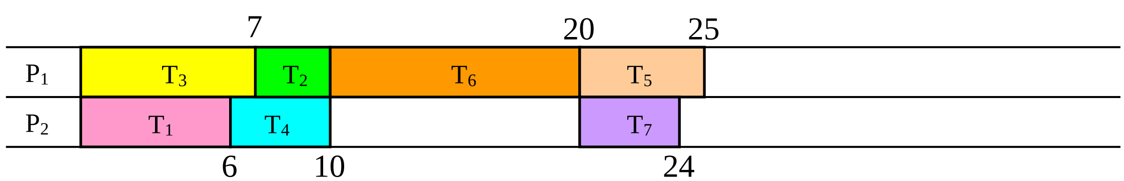
Time 10: Both processors complete their tasks. T_6 becomes ready, and is assigned to P_1. No other tasks are ready, so P_2 idles.

Priority list: ~~T_6,~~ ~~T_3,~~ T_{10}, ~~T_1,~~ T_5, ~~T_4,~~ T_7, ~~T_2,~~ T_8, T_9



Time 20: With T_6 complete, T_5 and T_7 become ready, and are assigned to P_1 and P_2 respectively.

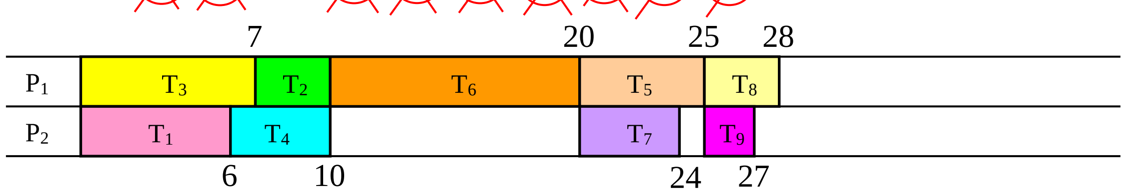
Priority list: ~~T_6,~~ ~~T_3,~~ T_{10}, ~~T_1,~~ ~~T_5,~~ ~~T_4,~~ ~~T_7,~~ ~~T_2,~~ T_8, T_9



Time 24: P_2 completes T_7. No new items become ready, so P_2 idles.

Time 25: P_1 completes T_5. T_8 and T_9 become ready, and are assigned.

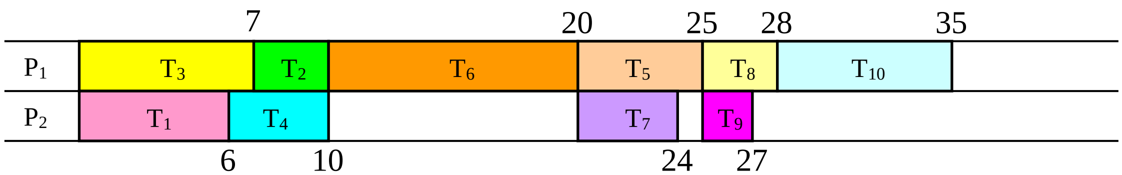
Priority list: ~~T_6,~~ ~~T_3,~~ T_{10}, ~~T_1,~~ ~~T_5,~~ ~~T_4,~~ ~~T_7,~~ ~~T_2,~~ ~~T_8,~~ ~~T_9~~



Time 27: T_9 is completed. No items ready, so P_2 idles.

Time 28: T_8 is completed. T_{10} becomes ready, and is assigned to P_1.

Priority list: ~~T_6~~, ~~T_3~~, ~~T_{10}~~, ~~T_1~~, ~~T_5~~, ~~T_4~~, ~~T_7~~, ~~T_2~~, ~~T_8~~, ~~T_9~~

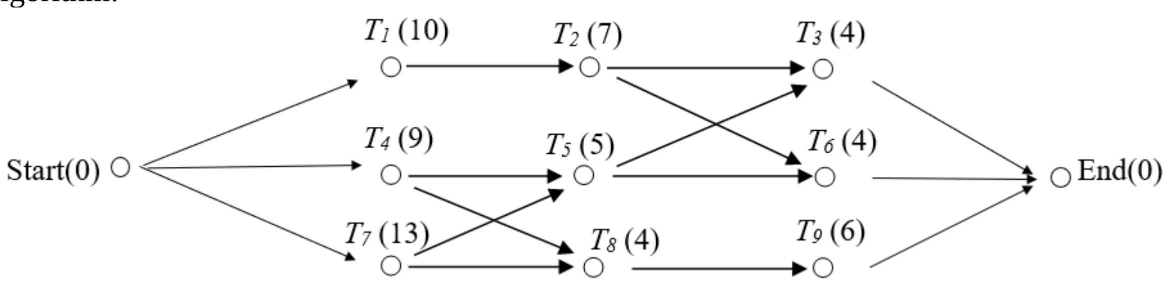


This is our completed schedule, with a finishing time of 35.

Using the decreasing time algorithm, the priority list led to a schedule with a finishing time of 35. Is this good? It certainly looks like there was a lot of idle time in this schedule. To get some idea how good or bad this schedule is, we could compute the critical time, the minimum time to complete the job. To find this, we look for the sequence of tasks with the highest total completion time. For this digraph that sequence would appear to be: T_2, T_6, T_5, T_8, T_{10}, with total sequence time of 28. From this we can conclude that our schedule isn't horrible, but there is a possibility that a better schedule exists.

Try it Now 2

Determine the priority list for the digraph from Try it Now 1 using the decreasing time algorithm:



Sec. 2.4 Critical Path Algorithm

A sequence of tasks in the digraph is called a **path**. In the previous example, we saw that the critical path dictates the minimum completion time for a schedule. Perhaps, then, it would make sense to consider the critical path when creating our schedule. For example, in the last schedule, the processors began working on tasks 1 and 3 because they were longer tasks, but starting on task 2 earlier would have allowed work to begin on the long task 6 earlier.

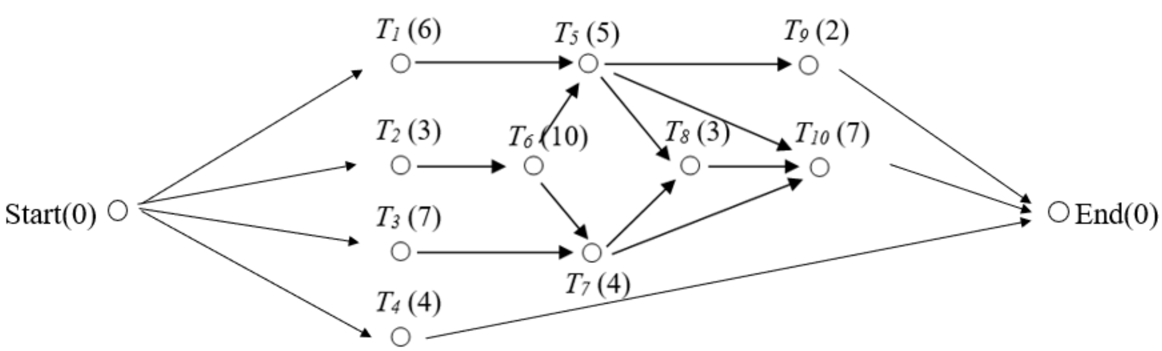
The critical path algorithm allows you to create a priority list based on idea of critical paths.

Critical Path Algorithm (version 1)

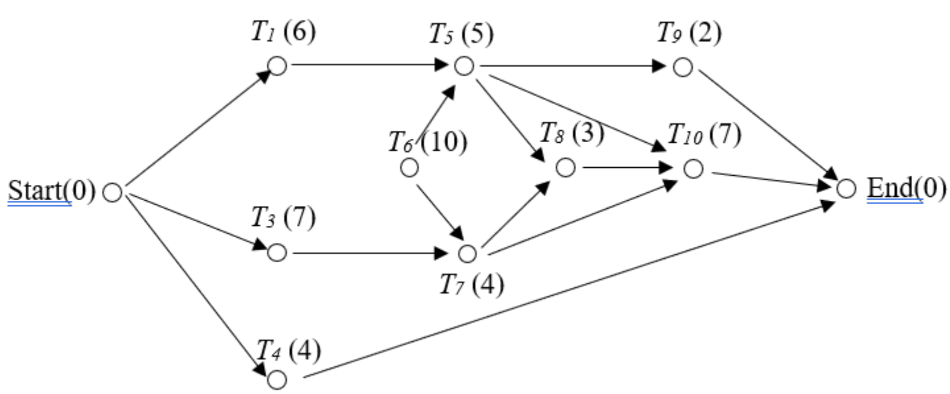
1. Find the critical path.
2. The first task in the critical path gets added to the priority list.
3. Remove that task from the digraph
4. Repeat, finding the new critical path with the revised digraph.

Example 4

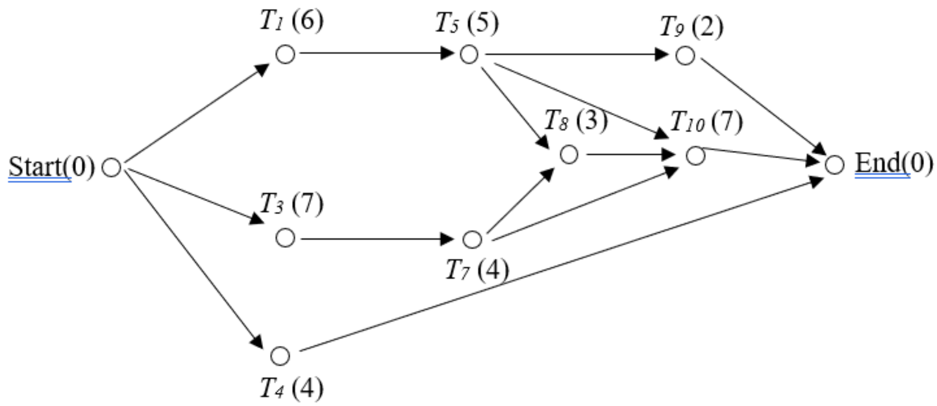
Recall the digraph from Example 3:



This digraph has critical path T_2, T_6, T_5, T_8, T_{10}, so T_2 gets added first to the priority list. Removing T_2 from the digraph, the digraph now looks like:



The critical path (longest path) in the remaining digraph is now T_6, T_5, T_8, T_{10}, so we add T_6 to the priority list and remove the vertex:



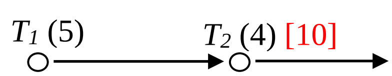
Now there are two paths with the same longest length: T_1, T_5, T_8, T_{10} and T_3, T_7, T_8, T_{10}. We can add T_1 to the priority list (or T_3 – we usually add the one with smaller subscript) and remove it, and continue the process. When we are finished, we have the priority list:

T_2, T_6, T_1, T_3, T_5, T_7, T_8, T_{10}, T_4, T_9.

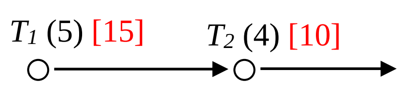
Searching for the critical path every time you remove a task from the digraph can get tedious tiring, especially for a large digraph. In practice, the critical path algorithm is implemented by first working from the end backwards. This is called the **backflow algorithm**.

Backflow Algorithm

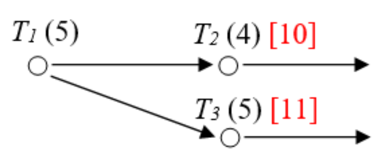
1. Introduce an "End" vertex, and assign it a time of 0, shown in [brackets]
2. Move backwards to every vertex that has an arrow to the end and assign it a critical time
3. From each of those vertices, move backwards and assign those vertices critical times. Notice that the critical time for the earlier vertex will be that task's time plus the critical time for the later vertex. For example,consider this segment of digraph.



In this case, if T_2 has already been determined to have a critical time of 10, then T_1 will have a critical time of 5+10 = 15



4. If you have already assigned a critical time to a vertex, replace it only if the new time is larger. For example, in the digraph below, T_1 should be labeled with a critical time of 16, since it is the longer of 5+10 and 5+11.



5. Repeat until all vertices are labeled with their critical times

Once you have completed the backflow algorithm, you can easily create the critical path priority list by using the critical times you just found.

Critical Path Algorithm (version 2)

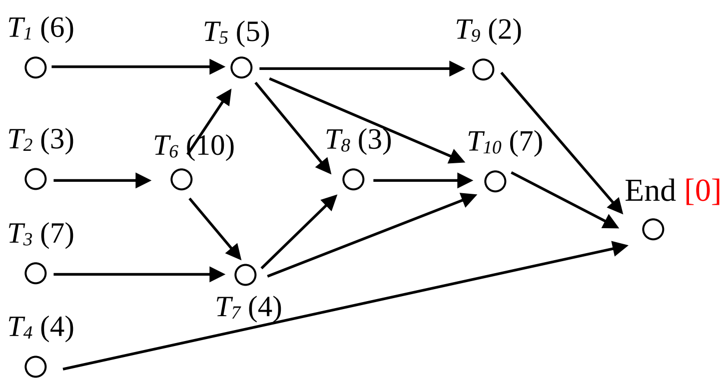
1. Apply the backflow algorithm to the digraph
2. Create the priority list by listing the tasks in order from longest critical time to shortest critical time

This version of the Critical Path Algorithm will usually be the easier to implement.

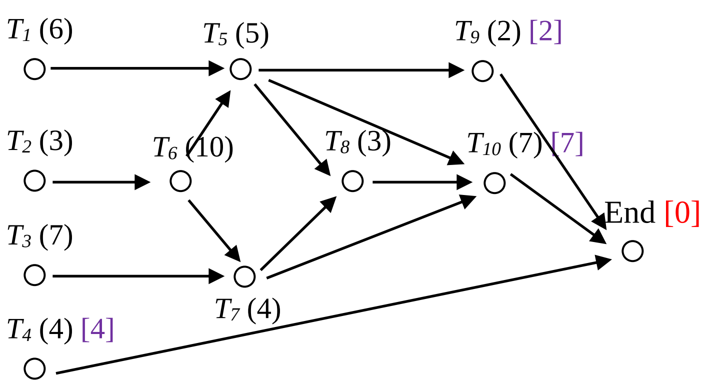
Example 5

Applying this to our digraph from the earlier example, we start by applying the backflow algorithm.

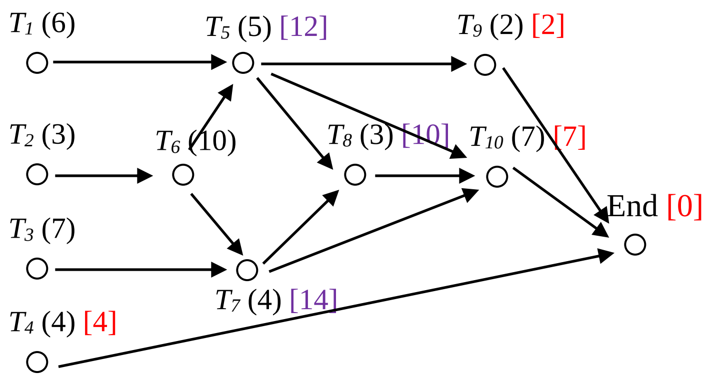
First, we add an end vertex and give it a critical time of 0:



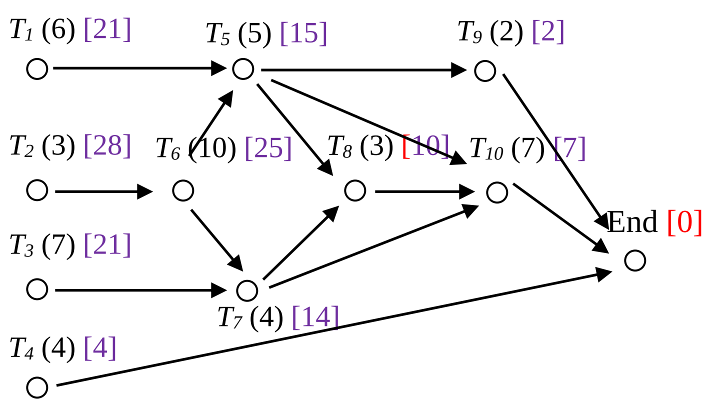
We then move back to T_4, T_9, and T_{10}, labeling them with their critical times



From each vertex marked with a critical time, we go back. T_7, for example, will get labeled with a critical time 11 – the task time of 4 plus the critical time of 7 for T_{10}. For T_5, there are two paths to the end. We use the longer, labeling T_5 with critical time 5+7 = 12.



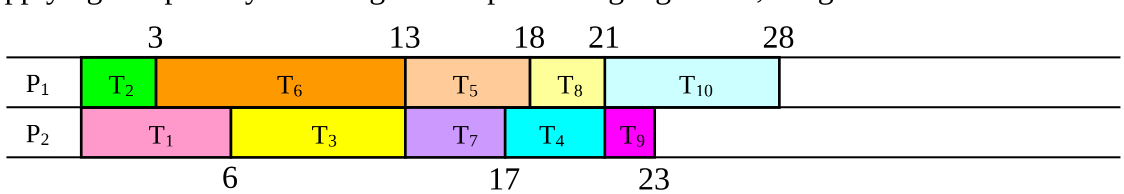
Continue the process until all vertices are labeled. Notice that the critical time for T_5 ended got replaced later with the even longer path through T_8.



We can now quickly create the critical path priority list by listing the tasks in decreasing order of critical time:

Priority list: T_2, T_6, T_1, T_3, T_5, T_7, T_8, T_{10}, T_4, T_9

Applying this priority list using the list processing algorithm, we get the schedule:



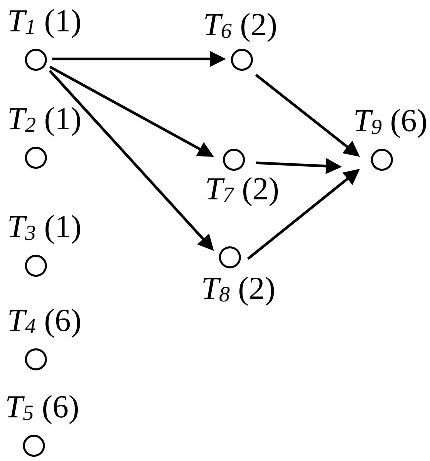
In this particular case, we were able to achieve the minimum possible completion time with this schedule, suggesting that this schedule is optimal. This is certainly not always the case.

Try it Now 3

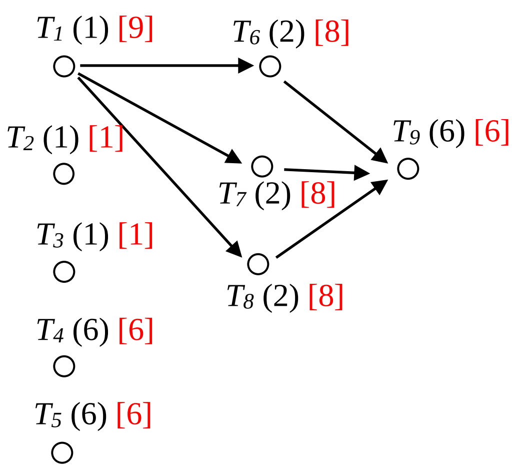
Determine the priority list for the digraph from Try it Now 1 using the critical path algorithm.

Example 6

This example is designed to show that the critical path algorithm doesn't always work wonderfully. Schedule the tasks in the digraph below on three processors using the critical path algorithm.

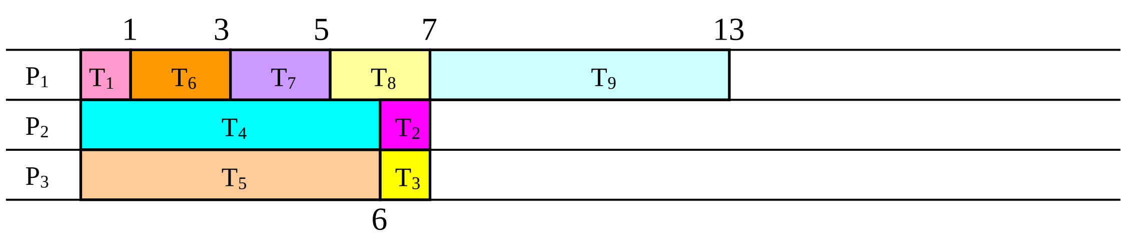


To create a critical path priority list, we could first apply the backflow algorithm:

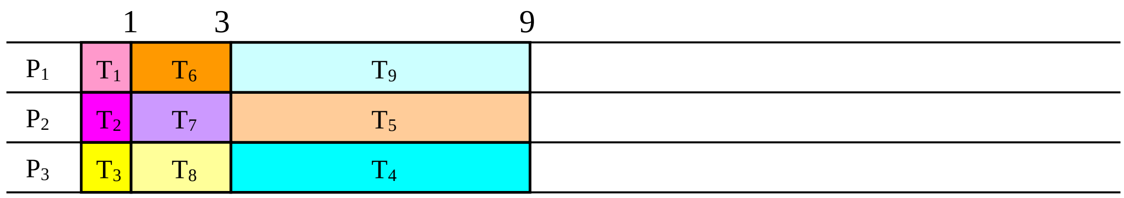


This yields the critical-path priority list: T_1, T_6, T_7, T_8, T_4, T_5, T_9, T_2, T_3.

Applying the list processing algorithm to this priority list leads to the schedule:



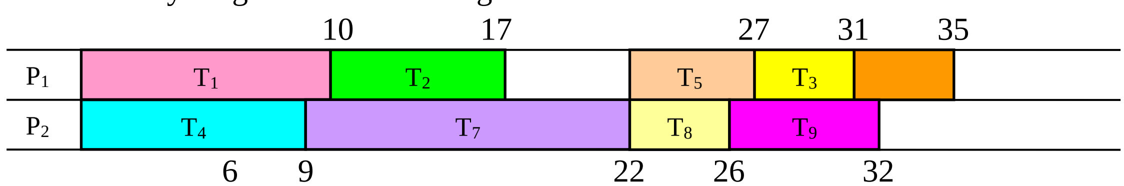
This schedule has finishing time of 13.
By observation, we can see that a much better schedule exists for the example above:



In most cases the critical path algorithm will lead to a very good schedule. There are cases, like this, where it will not. Unfortunately, there is no known algorithm to always produce the optimal schedule.

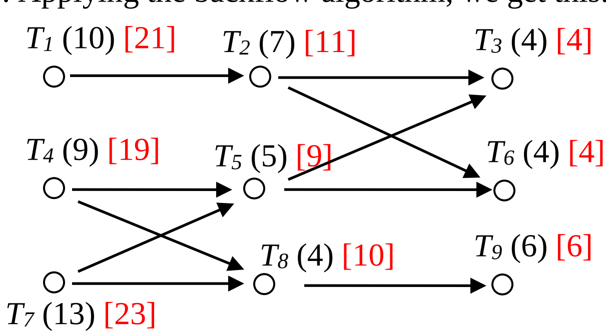
Try it Now Answers

1. Time 0: Tasks 1, 4, and 7 are ready. Assign Tasks 1 and 4
Time 9: Task 4 completes. Only task 7 is ready; assign task 7
Time 10: Task 1 completes. Task 2 now ready. Assign task 2
Time 17: Task 2 completes. Nothing is ready. Processor 1 idles
Time 22: Task 7 completes. Tasks 5 and 8 are ready. Assign tasks 5 and 8
Time 26: Task 8 completes. Task 9 is ready. Assign task 9
Time 27: Task 5 completes. Tasks 3 and 6 are ready. Assign task 3
Time 31: Task 3 completes. Assign task 6
Time 35: Everything is done. Finishing time is 35 for this schedule.



2. T_7, T_1, T_4, T_2, T_9, T_5, T_3, T_6, T_8

3. Applying the backflow algorithm, we get this:



The critical path priority list is: T_7, T_1, T_4, T_2, T_8, T_5, T_9, T_3, T_6

Exercises

Skills

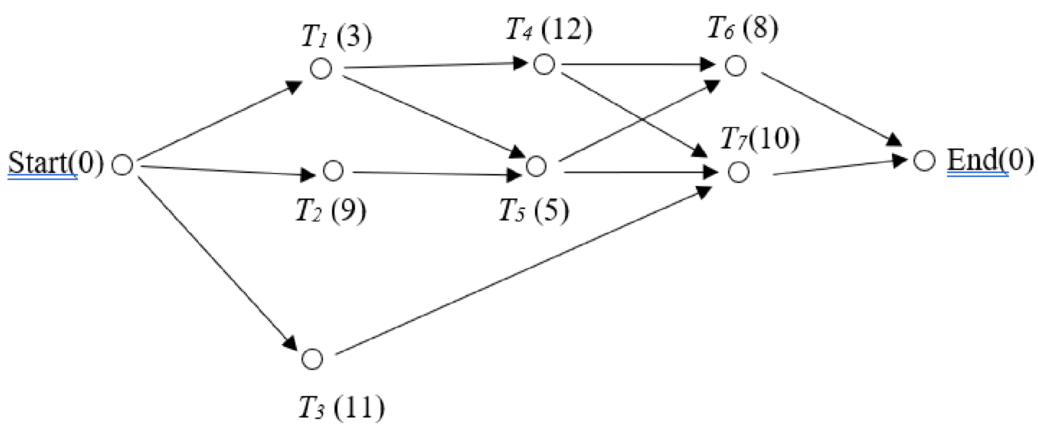
1. Create a digraph for the following set of tasks:

Task	Time required	Tasks that must be completed first
A	3	
B	4	
C	7	
D	6	A, B
E	5	B
F	5	D, E
G	4	E

2. Create a digraph for the following set of tasks:

Task	Time required	Tasks that must be completed first
A	3	
B	4	
C	7	
D	6	A
E	5	A
F	5	B
G	4	D, E

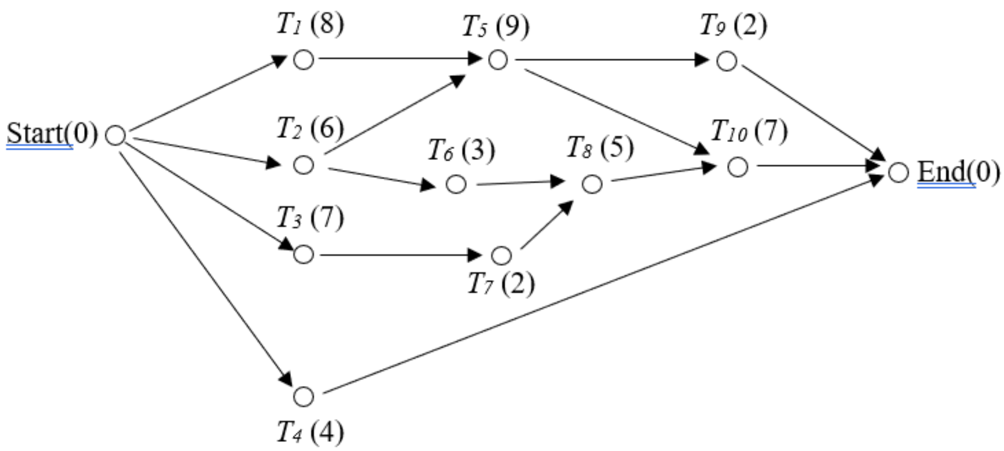
Use this digraph for the next 2 problems.



3. Using the priority list T_4, T_1, T_7, T_3, T_6, T_2, T_5, schedule the project with two processors.

4. Using the priority list T_5, T_2, T_3, T_7, T_1, T_4, T_6, schedule the project with two processors.

Use the digraph below for the next 4 problems:



5. Using the priority list T_4, T_3, T_9, T_{10}, T_8, T_5, T_6, T_1, T_7, T_2 schedule the project with two processors.

6. Using the priority list T_2, T_4, T_6, T_8, T_{10}, T_1, T_3, T_5, T_7, T_9 schedule the project with two processors.

7. Using the priority list T_4, T_3, T_9, T_{10}, T_8, T_5, T_6, T_1, T_7, T_2 schedule the project with three processors.

8. Using the priority list T_2, T_4, T_6, T_8, T_{10}, T_1, T_3, T_5, T_7, T_9 schedule the project with three processors.

9. Use the decreasing time algorithm to create a priority list for the digraph from #3, and schedule with two processors.

10. Use the decreasing time algorithm to create a priority list for the digraph from #3, and schedule with three processors.

11. Use the decreasing time algorithm to create a priority list for the digraph from #5, and schedule with two processors.

12. Use the decreasing time algorithm to create a priority list for the digraph from #5, and schedule with three processors.

13. Use the decreasing time algorithm to create a priority list for the problem from #1, and schedule with two processors.

14. Use the decreasing time algorithm to create a priority list for the problem from #2, and schedule with two processors.

15. Using the digraph from #3:
    a. Apply the backflow algorithm to find the critical time for each task
    b. Find the critical path for the project and the minimum completion time
    c. Use the critical path algorithm to create a priority list and schedule on two processors.

16. Using the digraph from #3, use the critical path algorithm to schedule on three processors.

17. Using the digraph from #5:
    a. Apply the backflow algorithm to find the critical time for each task
    b. Find the critical path for the project and the minimum completion time
    c. Use the critical path algorithm to create a priority list and schedule on two processors.

18. With the digraph from #5, use the critical path algorithm to schedule on three processors.

19. Use the critical path algorithm to schedule the problem from #1 on two processors.

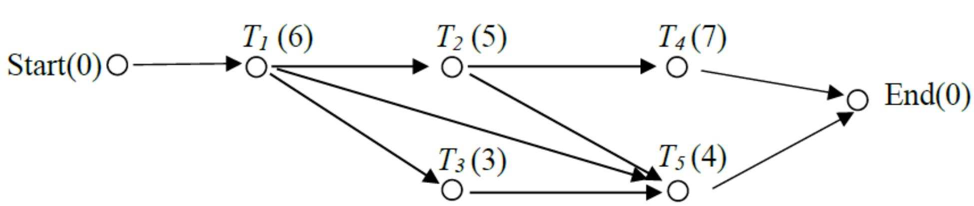
20. Use the critical path algorithm to schedule the problem from #2 on two processors.

Concepts

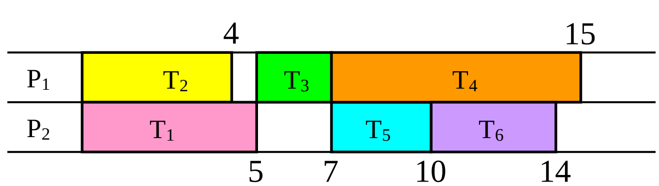
21. If an additional order requirement is added to a digraph, can the optimal finishing time ever become longer? Can the optimal finishing time ever become shorter?

22. Will an optimal schedule always have no idle time?

23. Consider the digraph below.
    a. How many priority lists could be created for these tasks?
    b. How many unique schedules are created by those priority lists?



24. Create a digraph and priority list that would lead to the schedule below.



25. Is it possible to create a digraph with three tasks for which every possible priority list creates a different schedule? If so, create it.

26. Is it possible to create a digraph with four tasks for which every possible priority list creates a different schedule? If so, create it.

Exploration

27. Independent tasks are ones that have no order requirements; they can be completed in any order.
    a. Consider three tasks, with completion times 2, 2, and 4 hours respectively. Construct two different schedules on two processors with different completion times to show that the priority list still matters with independent tasks.
    b. Choose a set of independent tasks with different completion times, and implement the decreasing time list algorithm and the critical path algorithm. What do you observe?
    c. Will using the decreasing time list or critical path algorithms with independent tasks always produce an optimal schedule? Why or why not?
    d. Will using the decreasing time list or critical path algorithms with independent tasks always produce the same schedule? Why or why not?

28. In a group, choose ten tasks necessary to throw a birthday party for a friend or child (for example, cleaning the house or buying a cake). Determine order requirements for the tasks, create a digraph, and schedule the tasks for two people.

29-37: At the end of the chapter it was noted that no algorithm exists to determine if an arbitrary schedule is optimal, but there are special cases where we can determine that a schedule is indeed optimal. In each of the following scenarios, determine:

a) If the scenario is even possible
b) Whether or not the schedule *could* be optimal
c) Whether or not we can be *sure* that the schedule is optimal

29. A job has a critical time of 30 hours, and the finishing time for the schedule on 2 processors is 30 hours.
30. The sum of all task times for a job is 40 hours, and the finishing time for the schedule on 2 processors was 15 hours
31. The sum of all task times for a job is 100 hours, the critical time of the job was 40 hours, and the finishing time for the schedule on 2 processors was 50 hours.
32. The sum of all task times for a job is 50 hours, and the finishing time for the schedule on 2 processors was 40 hours.
33. A job has a critical time of 30 hours, and the finishing time for the schedule on 3 processors is 20 hours.

34. The sum of all task times for a job is 60 hours, and the finishing time for the schedule on 3 processors was 20 hours.
35. The critical time for a job is 25 hours, and the finishing time for the schedule on 2 processors was 30 hours.
36. The sum of all task times for a job is 20 hours, and the finishing time for the schedule on 2 processors was 25 hours.
37. Based on your observations in the previous scenarios, write guidelines for when you can determine that a schedule is optimal.

CHAPTER 3
Probability

Excerpted from *Math in Society*, by David Lippman

Introduction

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

Sec. 3.1: Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

Solution: The sample space is the set of all possible simple events: {1,2,3,4,5,6}

Some examples of simple events:
  We roll a 1
  We roll a 5

Some compound events:
  We roll a number bigger than 4
  We roll an even number

Two dice

One die

Basic Probability
Given that all outcomes are equally likely, we can compute the probability of an event *E* using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally-likely outcomes}}$$

The definition above indicates that probabilities can naturally be written as fractions (sometimes reduced to lower terms), but they can also be written as decimals or percents.

Example 2

If we roll a 6-sided die, calculate

a) P(rolling a 1)
b) P(rolling a number bigger than 4)

Solution: Recall that the sample space is {1,2,3,4,5,6}

a) There is one outcome corresponding to "rolling a 1", so the probability is $\frac{1}{6}$

b) There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution: There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now 1

At some random moment, you look at your clock and note the minutes reading.

a. What is probability the minutes reading is 15?
b. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution: There are 52 cards in the deck and 4 Aces so $P(Ace) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – this occurs only if there are no outcomes that correspond with the event. The largest possible probability is 1 – this happens only if all possible outcomes correspond with the event.

Certain and Impossible events

An event that has a probability of 0 is called an *impossible event*.
An event that has a probability of 1 is called a *certain event*.
The probability of any event E must satisfy the inequality $0 \le P(E) \le 1$

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work*.

Sec. 3.2: Working with Events

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = 1/6$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining n - m outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event "*E* doesn't happen"

The notation $\bar{E}$ is used for the complement of event *E*.

We can compute the probability of the complement using $P\left(\bar{E}\right) = 1 - P(E)$

Notice also that $P(E) = 1 - P\left(\bar{E}\right)$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution: There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Independent Events

Example 6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution: We could list all possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}. Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was an example of two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 7

In each case, determine whether the events are independent:

a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

c) You draw a card from a deck, then draw a second card without replacing the first.

Solution:

a) The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

***P*(*A* and *B*) for independent events**

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Looking back at the coin and die example (Example 6), can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution: Observe that the type of tee shirt selected is independent of the sock selection.

The probability of choosing a white pair of socks is $\frac{6}{10}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

Thus, the probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 2

A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

In the previous examples, we were interested in the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution: Recall that the sample space is: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2}+\frac{1}{6}=\frac{6}{12}+\frac{2}{12}=\frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12}-\frac{1}{12}=\frac{7}{12}$.

We generalize this calculation to get the formula:

***P*(*A* or *B*)**
The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution: There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus, the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

When two events have not outcomes in common (as in the last example), we say that these events are **mutually exclusive**. For mutually exclusive events A and B, the formula simplifies to:

$$P(A \text{ or } B) = P(A) + P(B).$$

Example 11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution: Half the cards are red, so $P(\text{red}) = \frac{26}{52}$

There are four kings, so $P(\text{King}) = \frac{4}{52}$

There are two red kings, so $P(\text{Red and King}) = \frac{2}{52}$

We can then calculate:

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example 12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Find the probability that a randomly chosen person:

a) Has a red car *and* got a speeding ticket
b) Has a red car *or* got a speeding ticket.

Solution: We can see that 15 of the 665 people surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets plus 135 with red cars but no ticket plus 45 with a ticket but no red car, giving a total of 195 people. So the probability is $\frac{195}{665} \approx 0.2932$.

We also could have found this probability by:

P(had a red car) + P(got a speeding ticket) – P(had a red car and got a speeding ticket)

$$= \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}.$$

Sec. 3.3 Conditional Probability

It is often necessary to compute the probability of an event given that another event has occurred.

Example 13

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Solution: It might seem that you could use the formula for the probability of two independent events and simply multiply to get $\frac{4}{52}\cdot\frac{4}{52}=\frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

$$P(\text{ace on second draw} \mid \text{an ace on the first draw}).$$

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51}=\frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52}\cdot\frac{3}{51}=\frac{12}{2652}=\frac{1}{221}$.

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as

$$P(B \mid A)$$

This is read as "the probability of B given A"

Example 14

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution: These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 15

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Find the probability that a randomly chosen person:

a) Has a speeding ticket *given* they have a red car

b) Has a red car *given* they have a speeding ticket

Solution:

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$\text{P(ticket | red car)} = \frac{15}{150} = \frac{1}{10} = 0.1$$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$\text{P(red car | ticket)} = \frac{15}{60} = \frac{1}{4} = 0.25.$$

This example shows that, in general, P(B | A) is **not** equal to P(A | B).

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Conditional Probability Formula

If Events A and B are not independent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

Example 16

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution: The probability that the first card is a spade is $\frac{13}{52}$. The

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$.

Thus, the probability that both cards are spades is $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Example 17

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution: We can satisfy this condition in two ways:

Case A) We can get the Ace of Diamonds first and then a black card or
Case B) We can get a black card first and then the Ace of Diamonds.

We first calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$.

Now for Case B: the probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is P(A) + P(B) - P(A and B). In this problem, P(A and B) = 0 since the first card cannot be the Ace of Diamonds and be a black card. Therefore, the probability of Case A or Case B is $\frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Try it Now 4

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Example 18

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results.

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

If a woman who used the test is chosen at random, find the probabilities:

a) P(not pregnant | positive test result)
b) P(positive test result | not pregnant)

Solution:

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.

=> P(not pregnant | positive test result) = $\frac{5}{75} \approx 0.067$.

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

=> P(positive test result | not pregnant) = $\frac{5}{19} \approx 0.263$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

Sec. 3.4: Bayes Theorem

In this section we concentrate on the more complex conditional probability problems we began looking at in the last section.

Example 19

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution: There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write P(disease)=0.001.)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say P(positive | disease)=1.)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that P(positive | no disease)=0.05.)

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute P(disease | positive). We already know that P(positive | disease)=1, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, 1/1000 of 1000 people is 1. Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect (0.05)(999)=49.95 (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing P(disease | positive). There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$\mathrm{P(disease \mid positive)} \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and (0.05)(99,900)=4995 test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$\mathrm{P(disease \mid positive)} \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A})}$$

In our earlier example, this translates to

$$P(\text{disease} \mid \text{positive}) = \frac{P(\text{disease})P(\text{positive} \mid \text{disease})}{P(\text{disease})P(\text{positive} \mid \text{disease}) + P(\text{no disease})P(\text{positive} \mid \text{no disease})}$$

Plugging in the numbers gives

$$P(\text{disease} \mid \text{positive}) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example 20

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Solution: Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{disease} \mid \text{positive}) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{disease} \mid \text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

Try it Now 5

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

Sec. 3.5: Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 21

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus, you could go to the restaurant 15 nights in a row and have a different meal each night.

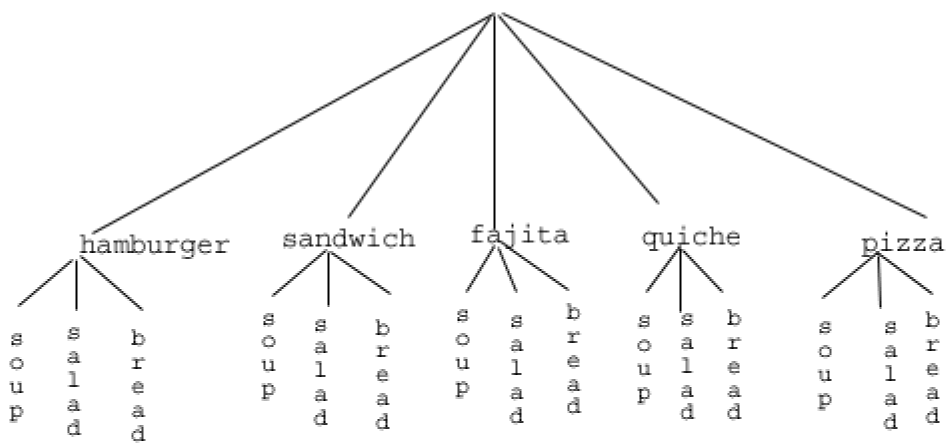
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	**sandwich**	**quiche**	**fajita**	**pizza**
soup	soup+burger				
salad	salad+burger				
bread	*etc.*				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose form instead of just two.

So, back to the problem in the example. What else can we do?
Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.

This is often called the **Fundamental Principle of Counting**.

Example 22

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 23

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution: There are 3 choices for an appetizer, 5 for the main course and 2 for dessert. So, there are a total of

$$3 \cdot 5 \cdot 2 = 30 \text{ possibilities.}$$

Example 24

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution: There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3, which is much more compact.

Try it Now 6

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 25

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution: This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code worth with the letters MATH.

In this example, we needed to calculate a product of the type $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics, and so we have a term for it. This is a **factorial**, and is denoted as $n!$

Factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

In other words, $n!$ is an abbreviation for the product of the first n positive integers.

Example 26

How many ways can five different door prizes be distributed among five people?

Solution: There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 27

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution: Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are $25 \cdot 24 \cdot 23 = 13{,}800$ ways in which the prizes can be awarded.

Example 28

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution: Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \cdot 7 \cdot 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the $100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities, we call this a **permutation of n objects, taken r at a time.**
The total number of such arrangements will be given by the formula

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, it is because there must be a total of r factors being multiplied (we are making r choices). Think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, and we easily verify that $n - r + 1 = 25 - 3 + 1 = 23$ is the last factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often; we only developed it so that we could attach a special notation and a special definition to situation where we are counting permutations. I.e. where we are choosing r items *without replacement* from a total of n possibilities and where the *order of selection matters*.

Permutations

A *permutation* is an **ordered** set of *r* objects, selected without replacement from an *n*- element set. The total number of permutations of *n* objects, taken *r* at a time is given by:

$$_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

We can express this result more simply using factorials:

$$_nP_r = \frac{n!}{(n-r)!}$$

Once we recognize that we are counting permutations, we can usually use technology rather than factorials or repeated multiplication to compute permutations. For example, the formula above is built into the TI-83/84 family of calculators, as well as programs like Excel.

Example 29

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution: Since we are choosing 4 paintings out of 9 *without replacement* and the *order of selection is important*, we are counting permutations. The total number of ways to display the paintings is given by:

$$_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3{,}024.$$

Example 30

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution: We want to choose 4 people out of 16 without replacement and the order of selection is important. So we are again counting permutations. The total number of ways the committee can be selected is

$$_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43{,}680.$$

Try it Now 7

How many 5 character passwords can be made using the letters A through Z

a. if repeats are allowed

b. if no repeats are allowed

Combinations

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 31

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution: Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \cdot 24 \cdot 23 =$ 13,800 ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \cdot 2 \cdot 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be 13,800/6 = 2300.

We can generalize the situation from the example above. Whenever we are choosing a collection of r items *without replacement* from a set of n objects, and the *order of selection is **not** important*, we call that a **combination of *n* objects, taken *r* at a time**.

Generalizing from the previous example, the number of such unordered arrangements is given by

$$\frac{{}_nP_r}{{}_rP_r} = {}_nP_r / r!$$

To summarize:

Combinations

A *combination* is an **unordered** set of r objects, selected without replacement from an n- element set. The total number of combinations of n objects, taken r at a time is

$$_nC_r = \frac{_nP_r}{_rP_r}$$

We can rewrite the combinations formula in terms of factorials:

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Example 32

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution: Since we are choosing 4 people out of 35 *without replacement* and the *order of selection does **not** matter,* there are

$$_{35}C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52{,}360$$

ways to choose the students.

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 33

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats.

How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution: In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10}$ = 3003 ways to choose the 10 Republicans and ${}_{14}C_{9}$ = 2002 ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6{,}012{,}006$ possible ways of selecting the members of the Defense Subcommittee.

Section 3.6: Probability and Counting

We can use the basic counting rule, permutations and combinations to help us answer more complex probability questions.

Example 34

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution: There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_{4}$ = 5040.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is

$$\frac{{}_{10}P_{4}}{10^4} = \frac{5040}{10000} = 0.504$$

Example 35

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn’t matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution: In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player’s ticket could match the six numbers drawn from the machine.

Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$P(\text{Win}) = \frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

Example 36

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution: As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6$ = 12,271,512. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers.

The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$
The number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$.

Thus, the number of favorable outcomes is then given by the Basic Counting Rule:

$${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252.$$

So, the probability of winning the second prize is:

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

Try it Now 9

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

In many card games (such as poker) the order in which the cards are drawn is not important, since the player may rearrange the cards in his hand any way he chooses. in the problems that follow, we will assume that this is the case unless otherwise stated.

Example 37

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution: The order in which the cards are dealt does not matter, so we use combinations to compute the possible number of 5-card hands. The total number is ${}_{52}C_5$.
This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck.

Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace
Since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces.

By the Basic Counting Rule, there are ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 38

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution: The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 10
Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

> Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 39

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution: There are a lot of ways there could be at least one shared birthday. Fortunately, there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

So we start by computing the probability that there is no shared birthday. Imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (we ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 40

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution: Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365}\cdot\frac{364}{365}\cdot\frac{363}{365}\cdot\frac{362}{365}\cdot\frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 41

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution: Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: http://statweb.stanford.edu/~susan/surprise/Birthday.html, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 11
Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Section 3.7: Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 42

[1]In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Solution: Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, in total you've lost \$2. On a per-space basis, you have "won" -\$2/\$38 ≈ -\$0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

[1] Photo CC-BY-SA http://www.flickr.com/photos/stoneflower/

Notice that if we multiply each outcome by its corresponding probability we get

$$\$35 \cdot \frac{1}{38} = 0.9211 \text{ and } -\$1 \cdot \frac{37}{38} = -0.9737,$$

and if we add these numbers we get

$$0.9211 + (-0.9737) \approx -0.053,$$

which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

Try it Now 12

You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

Example 43

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Solution: Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1-\frac{253}{12271512}=\frac{12271259}{12271512}$

Thus, the expected value is:

$$(\$999{,}999)\cdot\frac{1}{12271512}+(\$999)\cdot\frac{252}{12271512}+(-\$1)\cdot\frac{12271259}{12271512}\approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now 13

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 44

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year[2]. An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

Solution: The probabilities and outcomes are

Outcome	Probability of outcome
$100,000 - $275 = $99,725	0.00242
-$275	1 – 0.00242 = 0.99758

The expected value is ($99,725)(0.00242) + (-$275)(0.99758) = -$33.

[2] According to the estimator at http://www.numericalexample.com/index.php?view=article&id=91

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

Try it Now Answers

1. There are 60 possible readings, from 00 to 59. a. $\frac{1}{60}$ b. $\frac{16}{60}$ (counting 00 through 15)

2. Since the second draw is made after replacing the first card, these events are independent. The probability of an ace on each draw is $\frac{4}{52}=\frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$

3. P(white sock and white tee) = $\frac{6}{10}\cdot\frac{3}{7}=\frac{9}{35}$

   P(white sock or white tee) = $\frac{6}{10}+\frac{3}{7}-\frac{9}{35}=\frac{27}{35}$

4. a. $\frac{6}{10}\cdot\frac{5}{9}=\frac{30}{90}=\frac{1}{3}$

5. Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

   P(disease | positive) = $\frac{500}{500+2985}=\frac{500}{3485}\approx 14.3\%$

6. $8\cdot 11\cdot 5 = 440$ menu combinations

7. There are 26 characters. a. $26^5 = 11{,}881{,}376$. b. ${}_{26}P_5 = 26\cdot25\cdot24\cdot23\cdot22 = 7{,}893{,}600$

8. Order does not matter. ${}_{29}C_{19} = 20{,}030{,}010$ possible subcommittees

9. There are $5^{10} = 9{,}765{,}625$ different ways the exam can be answered. There are 10 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 40 ways the test could be answered with one wrong answer.

   P(9 answers correct) = $\frac{40}{5^{10}}\approx 0.0000041$ chance

10. $P(\text{three Aces and two Kings}) = \dfrac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \dfrac{24}{2598960} \approx 0.0000092$

11. $P(\text{shared birthday}) = 1 - \dfrac{{}_{365}P_{10}}{365^{10}} \approx 0.117$

12. $(\$3{,}995)\cdot\dfrac{1}{2000} + (-\$5)\cdot\dfrac{1999}{2000} \approx$ -\$3.00

13. Suppose you roll the first die. The probability the second will be different is $\dfrac{5}{6}$. The probability that the third roll is different than the previous two is $\dfrac{4}{6}$, so the probability that the three dice are different is $\dfrac{5}{6}\cdot\dfrac{4}{6} = \dfrac{20}{36}$.

The probability that two dice will match is the complement, $1 - \dfrac{20}{36} = \dfrac{16}{36}$.

The expected value is: $(\$2)\cdot\dfrac{16}{36} + (-\$1)\cdot\dfrac{20}{36} = \dfrac{12}{36} \approx \$0.33$.

Yes, it is in your advantage to play. On average, you'd win \$0.33 per play.

Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
    a. A red ball is drawn
    b. A white ball is drawn

2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
    a. A consonant
    b. A vowel

3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.

4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.

5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.

6. Compute the probability of tossing a six-sided die and getting a 7.

7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.

10. Compute the probability of tossing a six-sided die and getting a number less than 3.

11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?

12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?

13. Compute the probability of rolling a 12-sided die and getting a number other than 8.

14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?

15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?

16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?

17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?

18. A fair coin is flipped twice. What is the probability of showing heads on both flips?

19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?

20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?

21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.

22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.

23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
    a. Bert and Ernie both draw an Ace.
    b. Bert draws an Ace but Ernie does not.
    c. neither Bert nor Ernie draws an Ace.
    d. Bert and Ernie both draw a heart.
    e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.

24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
    a. Bert gets a Jack and Ernie rolls a five.
    b. Bert gets a heart and Ernie rolls a number less than six.
    c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
    d. Bert gets a red card and Ernie rolls a fifteen.
    e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.

26. Compute the probability of drawing two spades from a deck of cards.

27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then a female.
    b. a female is selected, then a male.
    c. two males are selected.
    d. two females are selected.
    e. no males are selected.

28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then two females.
    b. a female is selected, then two males.
    c. two females are selected, then one male.
    d. three males are selected.
    e. three females are selected.

29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

31. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.

32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.

33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.

34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.

35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.

36. Compute the probability of drawing a King or a heart from a deck of cards.

37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Even-numbered given that the marble is red.
    b. Red given that the marble is even-numbered.

38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Odd-numbered given that the marble is blue.
    b. Blue given that the marble is odd-numbered.

39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.

40. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.

41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.

42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.

43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A | not B).

44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A | not B).

45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.

46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?

48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?

49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?

50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?

51. How many three-letter "words" can be made from 4 letters "FGHI" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed

52. How many four-letter "words" can be made from 6 letters "AEBWDP" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed

53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?

54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?

55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?

56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?

57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?

59. In western music, an octave is divided into 12 pitches. For the film *Close Encounters of the Third Kind*, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?

60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?

61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?

62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?

63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?

64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?

65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?

66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?

67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?

68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.

69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?

70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?

71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.

72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win \$3. If it is silver, you win \$2. If it is black, you lose \$1. What is your expected value if you play this game?

74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you \$3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him \$1. Compute the expected value for this game. Should you play this game?

75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose \$1. Find the expected value of this game.

76. A game is played by picking two cards from a deck. If they are the same value, then you win \$5, otherwise you lose \$1. What is the expected value of this game?

77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value of each warranty sold?

78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be \$60,000. If the company offers earthquake insurance for \$100, what is their expected value of the policy?

Exploration

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.
    a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
    b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?

Department	Men		Women	
	Applied	Admitted	Applied	Admitted
Dept A	400	240	100	90
Dept B	100	15	600	150

    c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?

80. A bet on "black" in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from \$1 to \$100 on each spin.
    a. Suppose you have \$10, and are going to play until you go broke or have \$20. What is your best strategy for playing?
    b. Suppose you have \$10, and are going to play until you go broke or have \$30. What is your best strategy for playing?

81. Your friend proposes a game: You flip a coin. If it's heads, you win \$1. If it's tails, you lose \$1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?

82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?

83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?

84. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "ALICE"
    b. "APPLE"

85. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "TRUMPS"
    b. "TEETER"

86. The *Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?

87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails

88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails

89. In this problem, we will explore probabilities from a series of events.
    a. If you flip 20 coins, how many would you *expect* to come up "heads", on average? Would you expect *every* flip of 20 coins to come up with exactly that many heads?
    b. If you were to flip 20 coins, what would you consider a "usual" result? An "unusual" result?
    c. Flip 20 coins (or one coin 20 times) and record how many come up "heads". Repeat this experiment 9 more times. Collect the data from the entire class.
    d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
    e. Based on the class's experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
    f. The formula ${}_nC_x p^x(1-p)^{n-x}$ will compute the probability of an event with probability p occurring x times out of n, such as flipping x heads out of n coins where the probability of heads is $p = ½$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
    g. If you were to flip 20 coins, based on the class's experimental data, what range of values would you consider a "usual" result? What is the combined probability of these results? What would you consider an "unusual" result? What is the combined probability of these results?
    h. We'll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

CHAPTER 4
SET THEORY

Excerpted from *Math in Society*, by David Lippman

Sets

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set
A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:
a) The set of all even numbers
b) The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:
a) {1, 3, 9, 12}
b) {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation
Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol $\in$ means "is an element of".

A set that contains no elements, { }, is called the **empty set** and is notated $\emptyset$

Example 2

Let A = {1, 2, 3, 4}

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.

If B is a subset of A, we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A, we write $B \subset A$

Example 3

Consider these three sets
A = the set of all even numbers $\quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A.

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A, since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays "Much Ado About Nothing", "MacBeth", and "A Midsummer's Night Dream". What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).
The union is notated $A \cup B$.
More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.
The intersection is notated $A \cap B$.
More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A.
The complement is notated A', or A^c, or sometimes $\sim A$.

Example 5

Consider the sets: A = {red, green, blue} B = {red, yellow, orange}
C = {red, orange, yellow, green, blue, purple}

a) Find $A \cup B$

The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange}
Notice we only list red once.

b) Find $A \cap B$

The intersection contains all the elements in both sets: $A \cap B$ = {red}

c) Find $A^c \cap C$

Here we're looking for all the elements that are *not* in set A and are also in C.
$A^c \cap C$ = {orange, yellow, purple}

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Notice that in the example above, it would be hard to just ask for A^c, since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A.

Example 6

a) If we were discussing searching for books, the universal set might be all the books in the library.
b) If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
c) If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is U = all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then

$A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}
W = {duck, rabbit, deer, frog, mouse}

a) Find $(H \cap F) \cup W$

We start with the intersection: $H \cap F$ = {dog, rabbit}
Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}

b) Find $H \cap (F \cup W)$

We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}
Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}

c) Find $(H \cap F)^c \cap W$

We start with the intersection: $H \cap F$ = {dog, rabbit}
Now we want to find the elements of W that are *not* in $H \cap F$
$(H \cap F)^c \cap W$ = {duck, deer, frog, mouse}

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

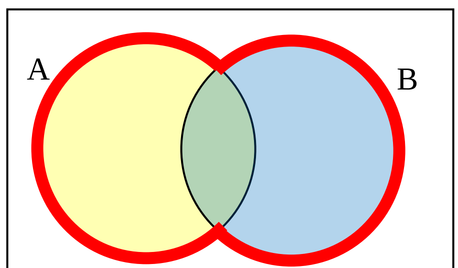
Venn Diagram
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

Example 9

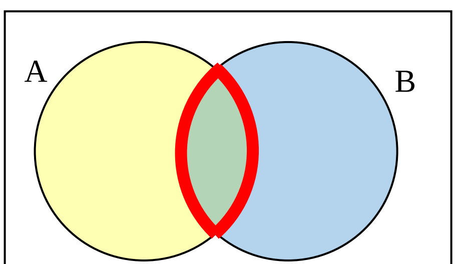
Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

$A \cup B$ contains all elements in *either* set.



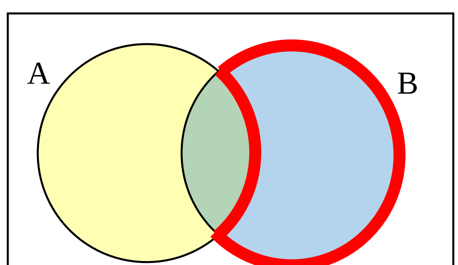
$A \cup B$

$A \cap B$ contains only those elements in both sets – in the overlap of the circles.



$A \cap B$

A^c will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.

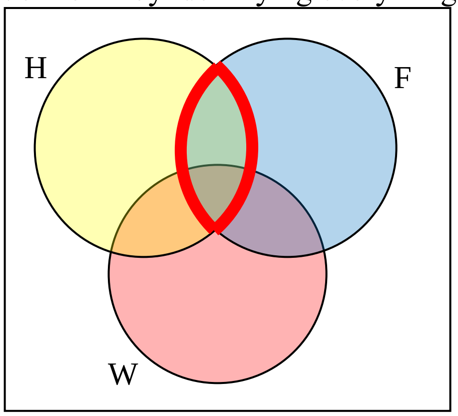


$A^c \cap B$

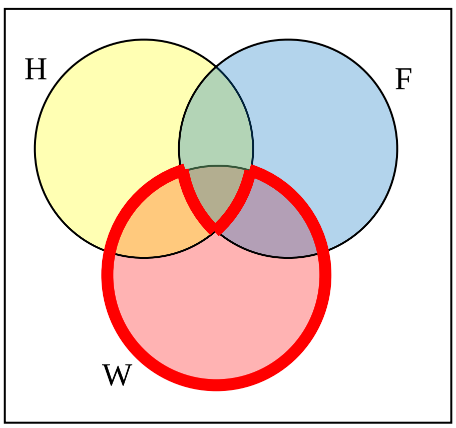
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$



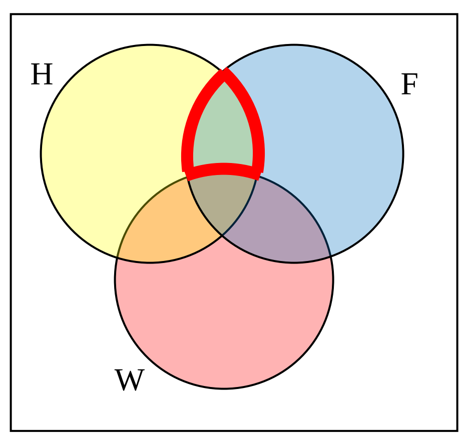
Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

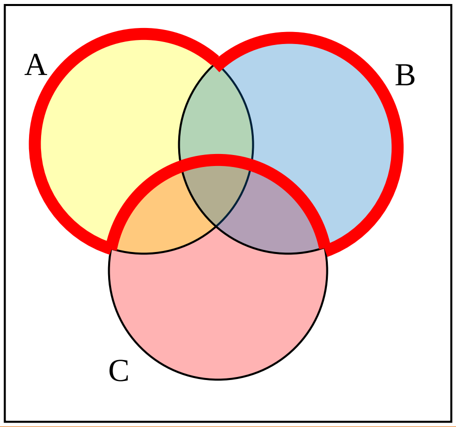
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set *are* in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$



Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown



Cardinality

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.
What is the cardinality of B? $A \cup B$, $A \cap B$?

The cardinality of B is 4, since there are 4 elements in the set.
The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.
The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of P = the set of English names for the months of the year?

The cardinality of this set is 12, since there are 12 months in the year.

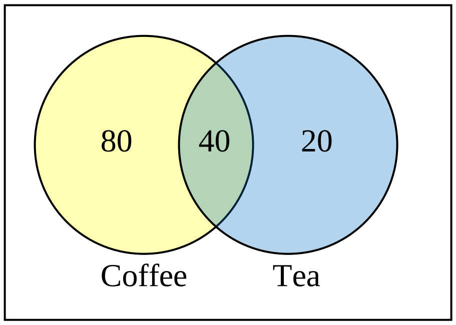
Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?



This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.
200 – 20 – 80 – 40 = 60 people who drink neither.

Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,
$n(F \cup T) = n(F) + n(T) - n(F \cap T)$
$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T$ = 90%, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A^c) = n(U) - n(A)$

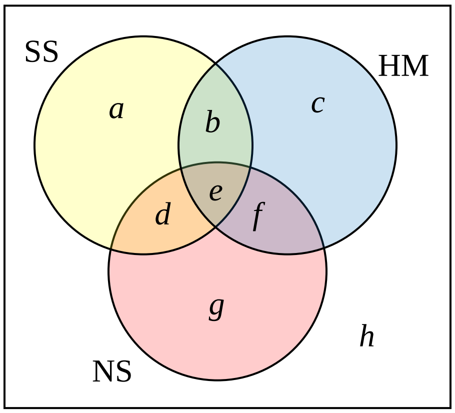
Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:
$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course	26 were taking a HM course
19 were taking a NS course	9 were taking SS and HM
7 were taking SS and NS	10 were taking HM and NS
3 were taking all three	7 were taking none

How many students are only taking a SS course?



It might help to look at a Venn diagram.
From the given data, we know that there are 3 students in region *e* and 7 students in region *h*.

Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be 7 – 3 = 4 students in region *d*.

Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be 10 – 3 = 7 students in region *f*.

Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b*.

Now, we know that 21 students were taking a SS course. This includes students from regions *a*, *b*, *d*, and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a*.

8 students are taking only a SS course.

Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

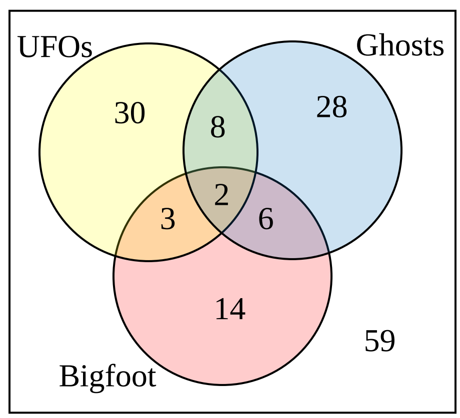
Try it Now Answers

1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2. $A \cup C$ = {red, orange, yellow, green, blue purple}
$B^c \cap A$ = {green, blue}

3. $A \cup B \cap C^c$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves 150 – 91 = 59 who believe in none.



Exercises

1. List out the elements of the set "The letters of the word Mississipi"

2. List out the elements of the set "Months of the year"

3. Write a verbal description of the set {3, 6, 9}

4. Write a verbal description of the set {a, i, e, o, u}

5. Is {1, 3, 5} a subset of the set of odd integers?

6. Is {A, B, C} a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.
$A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$ $U = \{\text{numbers from 0 to 10}\}$

7. $3 \in B$ 8. $5 \in C$ 9. $B \subset A$ 10. $C \subset A$ 11. $C \subset B$ 12. $C \subset U$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$ 14. $A \cup C$ 15. $A \cap C$ 16. $B \cap C$ 17. A^c 18. B^c

Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

19. $D^c \cap E$ 20. $F^c \cap D$ 21. $(D \cap E) \cup F$ 22. $D \cap (E \cup F)$

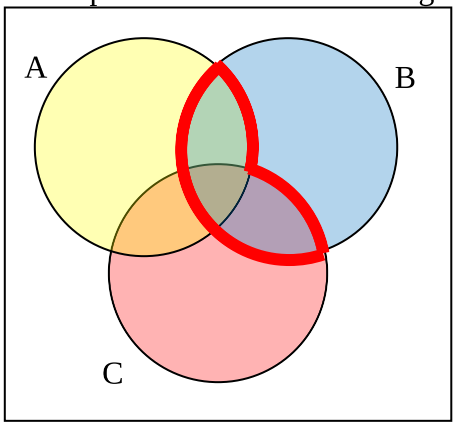
23. $(F \cap E)^c \cap D$ 24. $(D \cup E)^c \cap F$

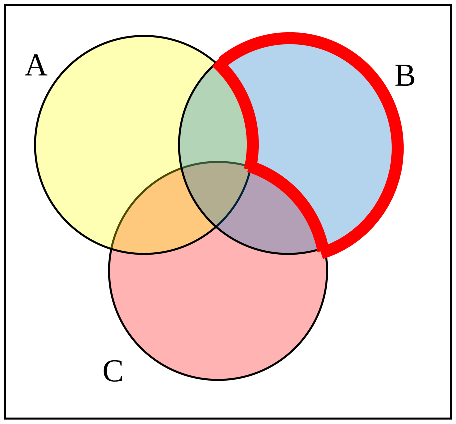
Create a Venn diagram to illustrate each of the following:

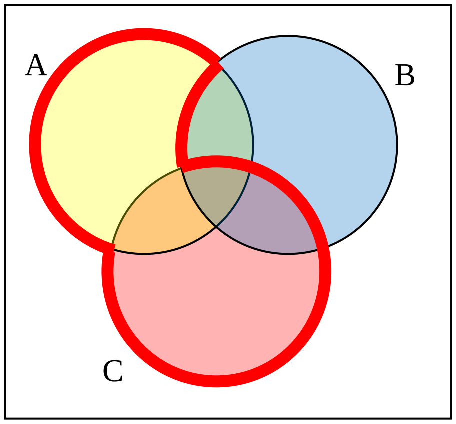
25. $(F \cap E) \cup D$ 26. $(D \cup E)^c \cap F$

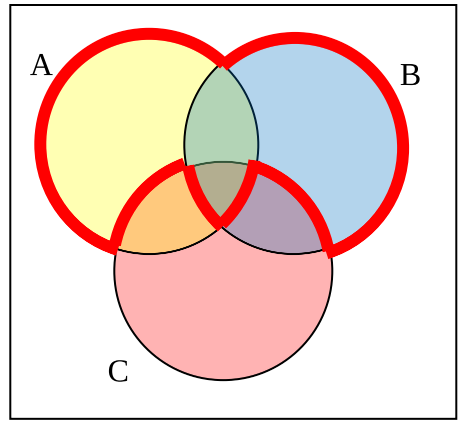
27. $(F^c \cap E^c) \cap D$ 28. $(D \cup E) \cup F$

Write an expression for the shaded region.

29. 

30. 

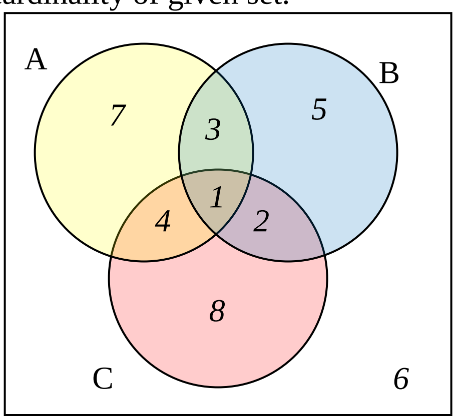
31. 

32. 

Let A = {1, 2, 3, 4, 5} B = {1, 3, 5} C = {4, 6}. Find the cardinality of the given set.

33. n(A) 34. n(B) 35. n($A \cup C$) 36. n($A \cap C$)

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. n($A \cap C$) 38. n($B \cup C$) 39. n($A \cap B \cap C^c$) 40. n($A \cap B^c \cap C$)

41. If n(G) = 20, n(H) = 30, n($G \cap H$) = 5, find n($G \cup H$)

42. If n(G) = 5, n(H) = 8, n($G \cap H$) = 4, find n($G \cup H$)

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix	62 only use Redbox
24 only use a video store	16 use only a video store and Redbox
48 use only Netflix and Redbox	30 use only a video store and Netflix
10 use all three	25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	8 only said size
16 only said brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen *The Matrix* (*M*)	24 had seen *Star Wars (SW)*
20 had seen *Lord of the Rings (LotR)*	10 had seen *M* and *SW*
14 had seen *LotR* and *SW*	12 had seen *M* and *LotR*
6 had seen all three	

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30% use the bus	20% ride a bicycle
25% walk	5% use the bus and ride a bicycle
10% ride a bicycle and walk	12% use the bus and walk
2% use all three	

CHAPTER 5
LOGIC

Excerpted from *Math in Society*, by David Lippman

Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like "and" and "or" – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

Sec. 5.1: Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like "and", "or", and "not" to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

> **Boolean Logic**
> Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.
>
> In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for "mystery", we are looking for all the books that are an element of the set M; the search is true for books that are in the set.

When we search for "mystery *and* comedy", we are looking for a book that is an element of both sets, in the intersection. If we were to search for "mystery *or* comedy", we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for "*not* comedy", we are looking for any book in the library that is not a comedy, the complement of the set C.

Connection to Set Operations

A and B	elements in the intersection $A \cap B$
A or B	elements in the union $A \cup B$
not A	elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks "do you want to go to the park or the movies?" they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant "would you like fries or a drink with that?" Answering "both, please" is an acceptable answer.

Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

We could start with the search "Mexico *and* university", but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:
Mexico *and* university *not* "New Mexico"

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google's search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:
Mexico university -"New Mexico"

Example 2

Describe the numbers that meet the condition:

even and less than 10 and greater than 0

The numbers that satisfy all three requirements are {2, 4, 6, 8}

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

Example 3

Describe the numbers that meet the condition:

odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

The first three conditions limit us to the set {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set {3, 5, 9, 15}

Notice that we would have gotten a very different result if we had written

(odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give {3, 9, 15}. When combined with the last condition, though, this set expands without limits:
{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, …}

Example 4

The English phrase "Go to the store and buy me eggs and bagels or cereal" is ambiguous; it is not clear whether the requestors is asking for eggs always along with either bagels or cereal, or whether they're asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:
Eggs and (bagels or cereal) means Option 1: Eggs and bagels, Option 2: Eggs and cereal
(Eggs and bagels) or cereal means Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

Conditional Statements

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false. A statement like $3 < 5$ is true; a statement like "a rat is a fish" is false. A statement like "$x < 5$" is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

Statements and Conditionals
A **statement** is either true or false.
A **conditional** is a compound statement of the form
"if p then q" or "if p then q, else s".

Example 5

In common language, an example of a conditional statement would be "If it is raining, then we'll go to the mall. Otherwise we'll go for a hike."

The statement "If it is raining" is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

Example 6

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets. In Excel, you can enter an expression like
=IF(A1<2000, A1+1, A1*2)

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2”.

Example 7

The expression =IF(A1>5, 2*A1, 3*A1) is used. Find the result if A1 is 3, and the result if A1 is 8.

This is equivalent to saying
If A1 > 5, then calculate 2*A1. Otherwise, calculate 3*A1

If A1 is 3, then the condition is false, since 3 > 5 is not true, so we do the alternate action, and multiply by 3, giving 3*3 = 9

If A1 is 8, then the condition is true, since 8 > 5, so we multiply the value by 2, giving 2*8=16

Example 8

An accountant needs to withhold 15% of income for taxes if the income is below $30,000, and 20% of income if the income is $30,000 or more. Write an expression that would calculate the amount to withhold.

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: 0.15*income. If the income is more than 30,000, we need to calculate 20% of the income: 0.20*income.

In words we could write “If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20”. In Excel, we would write:

=IF(A1<30000, 0.15*A1, 0.20*A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 9

A parent might say to their child “if you clean your room and take out the garbage, then you can have ice cream.”

Here, there are two simpler conditions:
1) The child cleaning her room
2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room *or* take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition "A1 < 300 and A1 > 100" in Excel, you would need to enter "AND(A1<300, A1>100)". Likewise, for the condition "A1=4 or A1=6" you would enter "OR(A1=4, A1=6)"

Example 10

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents. A certain tax credit applies if someone with no dependents earns less than $10,000, or if someone with dependents earns less than $20,000. Write a rule that describes this.

There are two ways the rule is met:
income is less than 10,000 *and* dependents is 0, *or*
income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as
(A1 < 10000 *and* A2 = 0) *or* (A1 < 20000 *and* A2 > 0)

In Excel's format, we'd write
IF(OR(AND(A1<10000, A2=0), AND(A1<20000, A2>0)), "you qualify", "you don't qualify")

Quantified Statements

Words that describe an entire set, such as "all", "every", or "none", are called **universal quantifiers** because that set could be considered a universal set. In contrast, words or phrases such as "some", "one", or "at least one" are called **existential quantifiers** because they describe the existence of at least one element in a set.

Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 11

Suppose your friend says "Everybody cheats on their taxes." What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 12

Suppose your friend says "One of these six cartons of milk is leaking." What is the minimum amount of evidence you would need to prove your friend wrong?

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend's statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement
The negation of "all A are B" is "at least one A is not B".
The negation of "no A are B" is "at least one A is B".
The negation of "at least one A is B" is "no A are B".
The negation of "at least one A is not B" is "all A are B".

Example 13

"Somebody brought a flashlight." Write the negation of this statement.

The negation is "Nobody brought a flashlight."

Example 14

"There are no prime numbers that are even." Write the negation of this statement.

The negation is "At least one prime number is even."

Try it Now 1
Write the negation of "All Icelandic children learn English in school."

Sec. 5.2: Symbolic Logic and Truth Tables

Before discussing truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\wedge$ is used for *and*:	A and B is notated $A \wedge B$
The symbol $\vee$ is used for *or*:	A or B is notated $A \vee B$
The symbol ~ is used for *not*:	not A is notated $\sim A$

You can remember the first two symbols by relating them to the shapes for the union and intersection. $A \wedge B$ would be the elements that exist in both sets, in $\mathrm{A} \cap \mathrm{B}$. Likewise, $A \vee B$ would be the elements that exist in either set, in $\mathrm{A} \cup \mathrm{B}$. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 15

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke".

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

a. $P \vee C$

b. $P \wedge C$

c. $\sim P$

d. $\sim(P \vee C)$

e. $P \wedge \sim C$

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 2

Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

Truth table
A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 16

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use *S* to designate "is a sectional", and *C* to designate "has a chaise".
A truth table for this situation would look like this:

S	*C*	*S* or *C*
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if *S* is true and *C* is also true, then the complex statement "*S* or *C*" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Basic truth tables

Conjunction

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

A	$\sim A$
T	F
F	T

Truth tables really become useful when we analyze more complex Boolean statements.

Example 17

Create a truth table for the statement $A \vee \sim B$

When we create the truth table, we need to list all the possible truth value combinations for *A* and *B*. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

A	*B*
T	T
T	F
F	T
F	F

After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for *A* and write the truth values for $\sim B$.

A	*B*	$\sim B$
T	T	F
T	F	T
F	T	F
F	F	T

Next we can find the truth values of $A \vee \sim B$, using the first and third columns.

A	*B*	$\sim B$	$A \vee \sim B$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

The truth table shows that $A \vee \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game <u>or</u> if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then $A \vee \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is <u>false</u> that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 3

Create a truth table for this statement: $\sim A \wedge B$

Example 18

Create a truth table for the statement $A \wedge \sim(B \vee C)$

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for *A*, *B*, and *C*. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates T, F, T, F... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $B \vee C$. Now we will temporarily ignore the column for *A* and focus on *B* and *C*, writing the truth values for $B \vee C$.

A	*B*	*C*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	*B*	*C*	$B \vee C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \vee C$, working off the $B \vee C$ column we just created. (Ignore the first three columns and simply negate the values in the $B \vee C$ column.)

A	*B*	*C*	$B \vee C$	$\sim(B \vee C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of *A and* $\sim(B \vee C)$. (Ignore the second, third, and fourth columns.)

A	*B*	*C*	$B \vee C$	$\sim(B \vee C)$	$A \wedge \sim(B \vee C)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 4
Create a truth table for this statement: $(\sim A \wedge B) \vee \sim B$

Truth Tables: Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional
A conditional is a logical compound statement in which a statement p, called the antecedent, implies a statement q, called the consequent.

A conditional is written as $p \rightarrow q$ and is translated as "if p, then q".

Example 19

The English statement "If it is raining, then there are clouds is the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consquent becomes irrelevant.

Example 20

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:
1) You pay for expedited shipping and receive the jersey by Friday
2) You pay for expedited shipping and don't receive the jersey by Friday
3) You don't pay for expedited shipping and receive the jersey by Friday
4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 21

A friend tells you "If you upload that picture to Facebook, you'll lose your job." Under what conditions can you say that your friend was wrong?

There are four possible outcomes:
1) You upload the picture and lose your job
2) You upload the picture and don't lose your job
3) You don't upload the picture and lose your job
4) You don't upload the picture and don't lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't say that their statement was wrong. Even if you didn't upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 22

Construct a truth table for the statement $(m \wedge {\sim}p) \rightarrow r$

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \wedge {\sim}p$.

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

m	p	r	${\sim}p$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

m	p	r	${\sim}p$	$m \wedge {\sim}p$
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \wedge {\sim}p$? This makes it a lot easier to read the conditional from left to right.

m	p	r	${\sim}p$	$m \wedge {\sim}p$	r	$(m \wedge {\sim}p) \rightarrow r$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false—the fourth row of the table—then the antecedent $m \wedge {\sim}p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \wedge {\sim}p) \rightarrow r$, consider this: let m = we order meatballs, p = we order pasta, and r = Rob is happy. The statement $(m \wedge {\sim}p) \rightarrow r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original conditional is	"if p, then q"	$p \rightarrow q$
The converse is	"if q, then p"	$q \rightarrow p$
The inverse is	"if not p, then not q"	$\sim p \rightarrow \sim q$
The contrapositive is	"if not q, then not p"	$\sim q \rightarrow \sim p$

Example 23

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

Equivalence

A conditional statement and its contrapositive are logically equivalent.
The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 24

Suppose this statement is true: "If I eat this giant cookie, then I will feel sick." Which of the following statements must also be true?

a. If I feel sick, then I ate that giant cookie.
b. If I don't eat this giant cookie, then I won't feel sick.
c. If I don't feel sick, then I didn't eat that giant cookie.

a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.

b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.

c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 5

"If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?
a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.
c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

a. You don't park here and you get a ticket.
b. You don't park here and you don't get a ticket.
c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

The Negation of a Conditional
The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

$$\sim(p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 25

Which of the following statements is equivalent to the negation of “If you don’t grease the pan, then the food will stick to it” ?

a. I didn’t grease the pan and the food didn’t stick to it.
b. I didn’t grease the pan and the food stuck to it.
c. I greased the pan and the food didn’t stick to it.

a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.

b. This is essentially the original statement with no negation; the “if…then” has been replaced by “and”.

c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 6
“If you go swimming less than an hour after eating lunch, then you will get cramps.” Which of the following statements is equivalent to the negation of this statement?
a. I went swimming more than an hour after eating lunch and I got cramps.
b. I went swimming less than an hour after eating lunch and I didn’t get cramps.
c. I went swimming more than an hour after eating lunch and I didn’t get cramps.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider “If you submit your hours today, then you will be paid next Friday.” What the payroll rep really means is “If you submit your hours today, then you will be paid next Friday, and if you don’t submit your hours today, then you won’t be paid next Friday.” The conditional statement if t, then p also includes the inverse of the statement: if not t, then not p. A more compact way to express this statement is “You will be paid next Friday *if and only if* you submit your timesheet today.” A statement of this form is called a **biconditional**.

Biconditional
A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as $p \leftrightarrow q$ and is translated as “p if and only if q”.

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p, then q <u>and</u> if q, then p. The double-headed arrow shows that the conditional statement goes from left to right <u>and</u> from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 26

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.
b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.

b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.

c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 7

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.
a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

Example 27

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Whenever we have three component statements, we start by listing all the possible truth value combinations for *A*, *B*, and *C*. After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for *C* and focus on *A* and *B*, writing the truth values for $A \vee B$.

A	*B*	*C*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	*B*	*C*	$A \vee B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of *C*. (Ignore the $A \vee B$ column and simply negate the values in the *C* column.)

A	*B*	*C*	$A \vee B$	$\sim C$
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of $(A \vee B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

A	*B*	*C*	$A \vee B$	$\sim C$	$(A \vee B) \leftrightarrow \sim C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project *A* or project *B* (or both, if you have the time). If you do one of the projects, you will not get a crummy review (*C* is for crummy). So $(A \vee B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project *A* or project *B*." Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A, B, and C are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A, B, and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

De Morgan's Laws

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4:30PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague a is available AND colleague b is available: $\sim(a \wedge b)$. This situation is equivalent to either colleague a NOT being available OR colleague b NOT being available: $\sim a \vee \sim b$.

De Morgan's Laws

The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".

$$\sim(p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$

The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".

$$\sim(p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$

Example 28

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

a. You did not get them flowers or did not get them candy.
b. You did not get them flowers and did not get them candy.
c. You got them flowers or got them candy.

a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
b. This statement is equivalent to the original; $\sim(f \vee c)$ is equivalent to $\sim f \wedge \sim c$.
c. This statement says that you got them something, but we know that you did not.

Try it Now 8
To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

Sec. 5.3: Analyzing Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types
An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 29

The argument "when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store" is an inductive argument.

The premises are:
I forgot my purse last week
I forgot my purse today

The conclusion is:
I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 30

The argument "every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M." is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating inductive arguments
An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments
A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

Evaluating Deductive Arguments with Euler Diagrams

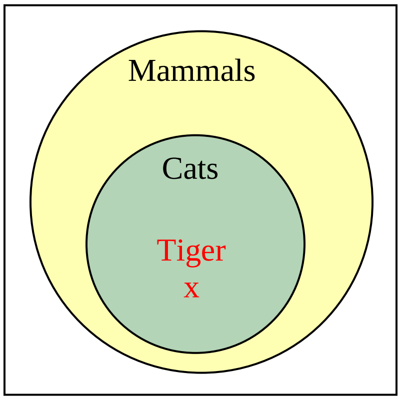
We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 31

Consider the deductive argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal." Is this argument valid?

The premises are:
All cats are mammals.
A tiger is a cat.

The conclusion is:
A tiger is a mammal.



Both the premises are true. To see that the premises must logically lead to the conclusion, we can use a Venn diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams

To analyze an argument with an Euler diagram:

1) Draw an Euler diagram based on the premises of the argument
2) The argument is invalid if there is a way to draw the diagram that makes the conclusion false
3) The argument is valid if the diagram cannot be drawn to make the conclusion false
4) If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

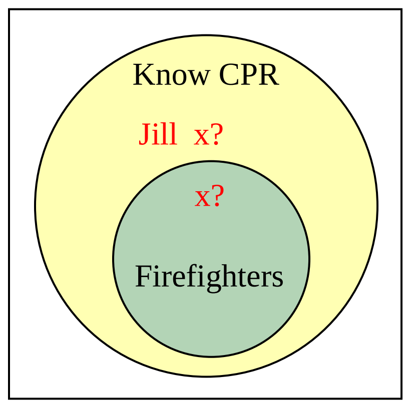
Try it Now 9

Determine the validity of this argument:

Premise: All cats are scared of vacuum cleaners.
Premise: Max is a cat.
Conclusion: Max is scared of vacuum cleaners.

Example 32

Premise: All firefighters know CPR.
Premise: Jill knows CPR.
Conclusion: Jill is a firefighter.



From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

Try it Now 10

Determine the validity of this argument:

Premise: All bicycles have two wheels.
Premise: This Harley-Davidson has two wheels.
Conclusion: This Harley-Davidson is a bicycle.

Try it Now 11

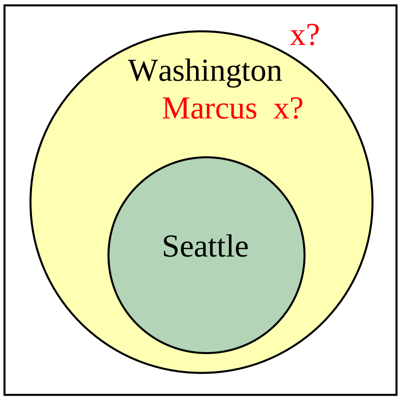
Determine the validity of this argument:

Premise: No cows are purple.
Premise: Fido is not a cow.
Conclusion: Fido is purple.

In addition to these categorical style premises of the form "all ___", "some ____", and "no ____", it is also common to see premises that are conditionals.

Example 33

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle.
Conclusion: Marcus does not live in Washington.



From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.

Try it Now 12

Determine the validity of this argument:

Premise: If you have lipstick on your collar, then you are cheating on me.
Premise: If you are cheating on me, then I will divorce you.
Premise: You do not have lipstick on your collar.
Conclusion: I will not divorce you.

Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 34

Consider the argument

Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let *b* represent "you bought bread" and *s* represent "you went to the store". Then the argument becomes:

Premise: $b \rightarrow s$
Premise: b
Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?

b	s	$b \rightarrow s$
T	T	T
T	F	F
F	T	T
F	F	T

b	s	$b \rightarrow s$	$(b \rightarrow s) \wedge b$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

b	s	$b \rightarrow s$	$(b \rightarrow s) \wedge b$	$[(b \rightarrow s) \wedge b] \rightarrow s$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

Try it Now 13

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.
Premise: I dug a hole.
Conclusion: Therefore, I had a shovel.

Example 35

Premise: If I go to the mall, then I'll buy new jeans.
Premise: If I buy new jeans, I'll buy a shirt to go with it.
Conclusion: If I go to the mall, I'll buy a shirt.

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.
The premises and conclusion can be stated as:
Premise: $m \rightarrow j$
Premise: $j \rightarrow s$
Conclusion: $m \rightarrow s$

We can construct a truth table for $[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$. Try to recreate each step and see how the truth table was constructed.

m	j	s	$m \rightarrow j$	$j \rightarrow s$	$(m \rightarrow j) \wedge (j \rightarrow s)$	$m \rightarrow s$	$[(m \rightarrow j) \wedge (j \rightarrow s)] \rightarrow (m \rightarrow s)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the final column of the truth table, we can see this is a valid argument.

Forms of Valid Arguments

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (*Modus Ponens*)
The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:
Premise: $p \rightarrow q$
Premise: p
Conclusion: q

The Latin name, *modus ponens*, translates to "mode that affirms".

Example 36

Recall this argument from an earlier example:
Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

In symbolic form:
Premise: $b \to s$
Premise: b
Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (*Modus Tollens*)
The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:
Premise: $p \to q$
Premise: $\sim q$
Conclusion: $\sim p$

The Latin name, *modus tollens*, translates to "mode that denies".

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \to \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 37

Premise: If I drop my phone into the swimming pool, my phone will be ruined.
Premise: My phone isn't ruined.
Conclusion: I didn't drop my phone into the swimming pool.

If we let d = I drop the phone in the pool and r = the phone is ruined, then we can represent the argument this way:
Premise $d \to r$
Premise $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 14
Is this argument valid?
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: Your toothbrush is dry.
Conclusion: You didn't brush your teeth before bed.

The Transitive Property (Hypothetical Syllogism)
The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise: $p \rightarrow q$
Premise: $q \rightarrow r$
Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 38

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.
Premise: If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let r = committing a reckless foul, y = receiving a yellow card, and s = being suspended, then our argument looks like this:

Premise $r \rightarrow y$
Premise $y \rightarrow s$
Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 15
Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise: $p \vee q$
Premise: $\sim p$
Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \vee q$ and $\sim q$, and the conclusion p.

Example 39

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let d = I drive and t = I take the train, then the symbolic representation of the argument is:

Premise $d \vee t$
Premise $\sim d$
Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 16

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.
Premise: Alison did not give a 5-minute speech.
Conclusion: Alison wrote a 10-page paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise:	$p \rightarrow q$
Premise:	q
Conclusion:	p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 40

Premise:	If I drink coffee after noon, then I have a hard time falling asleep that night.
Premise:	I had a hard time falling asleep last night.
Conclusion:	I drank coffee after noon yesterday.

If we let c = I drink coffee after noon and h = I have a hard time falling asleep, then our argument looks like this:

Premise	$c \rightarrow h$
Premise	h
Conclusion:	c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

Try it Now 17

Is this argument valid?

Premise:	If you pull that fire alarm, you will get in big trouble.
Premise:	You got in big trouble.
Conclusion:	You must have pulled the fire alarm.

The Fallacy of the Inverse
The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise:	$p \rightarrow q$
Premise:	$\sim p$
Conclusion:	$\sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \rightarrow \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 41

Premise:	If you listen to the Grateful Dead, then you are a hippie.
Premise:	Sky doesn't listen to the Grateful Dead.
Conclusion:	Sky is not a hippie.

If we let g = listen to the Grateful Dead and h = is a hippie, then this is the argument:

Premise	$g \rightarrow h$
Premise	$\sim g$
Conclusion:	$\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 18
Is this argument valid?

Premise:	If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise:	Alexei did not trip an opponent.
Conclusion:	Alexei will not be assessed a 2-minute penalty.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 42

Solve the puzzle. In other words, find a logical conclusion from these premises.
All babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

Let b = is a baby, d = is despised, i = is illogical, and m = can manage a crocodile.
Then we can write the premises as:
$b \rightarrow i$
$m \rightarrow \sim d$
$i \rightarrow d$

Writing the second premise correctly can be a challenge; it can be rephrased as "If you can manage a crocodile, then you are not despised."

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$; that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$; that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 43

Premise:	If I work hard, I'll get a raise.
Premise:	If I get a raise, I'll buy a boat.
Conclusion:	If I don't buy a boat, I must not have worked hard.

If we let h = working hard, r = getting a raise, and b = buying a boat, then we can represent our argument symbolically:

Premise	$h \rightarrow r$
Premise	$r \rightarrow b$
Conclusion:	$\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$; if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 19

Is this argument valid?

Premise:	If I go to the party, I'll be really tired tomorrow.
Premise:	If I go to the party, I'll get to see friends.
Conclusion:	If I don't see friends, I won't be tired tomorrow.

Sec. 5.4: Logical Fallacies in Common Language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Ad hominem
An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example 44

"Jane says that whales aren't fish, but she's only in the second grade, so she can't be right."

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 45

"Jane says that whales aren't fish, but everyone knows that they're really mammals. She's so stupid."

This certainly isn't very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

Appeal to ignorance
This type of argument assumes something it true because it hasn't been proven false.

Example 46

"Nobody has proven that photo isn't of Bigfoot, so it must be Bigfoot."

Appeal to authority
These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person's opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 47

"A diet high in bacon can be healthy; Doctor Atkins said so."

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 48

"Jennifer Hudson lost weight with Weight Watchers, so their program must work."

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person's experience would.

Appeal to consequence
An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 49

"Humans will travel faster than light: faster-than-light travel would be beneficial for space travel."

False dilemma
A false dilemma argument falsely frames an argument as an "either or" choice, without allowing for additional options.

Example 50

"Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens."

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

Circular reasoning
Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 51

"I shouldn't have gotten a C in that class; I'm an A student!"

In this argument, the student is claiming that because they're an A student, though shouldn't have gotten a C. But because they got a C, they're not an A student.

Post hoc (post hoc ergo propter hoc)
A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example 52

"Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning."

Straw man
A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 53

"Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists"

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones' position.

Correlation implies causation
Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example 54

"Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown."

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Try it Now 20
Identify the logical fallacy in each of the arguments

a. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
b. Since the 1950s, both the atmospheric carbon dioxide level and obesity levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
c. The oven was working fine until you started using it, so you must have broken it.
d. You can't give me a D in the class because I can't afford to retake it.
e. The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair, so we shouldn't do it.

It may be difficult to identify one particular fallacy for an argument. Consider this argument: "Emma Watson says she's a feminist, but she posed for these racy pictures. I'm a feminist, and no self-respecting feminist would do that." Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

Try it Now Answers

1. At least one Icelandic child did not learn English in school.

2. $C \vee \sim S$

3.

A	B	$\sim A$	$\sim A \wedge B$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

4.

A	B	$\sim A$	$\sim A \wedge B$	$\sim B$	$(\sim A \wedge B) \vee \sim B$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

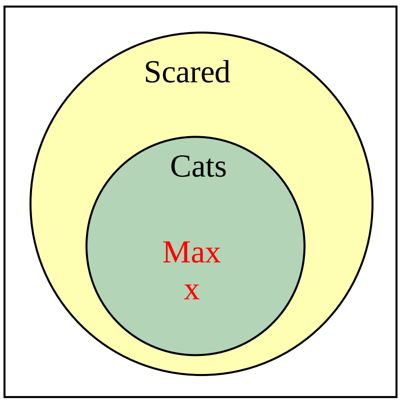
5. Choice b is correct because it is the contrapositive of the original statement.

6. Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

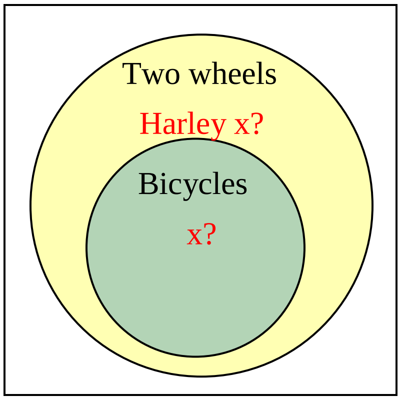
7. Choices a & b are false; c is true.

8. Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is “or” instead of “and”.

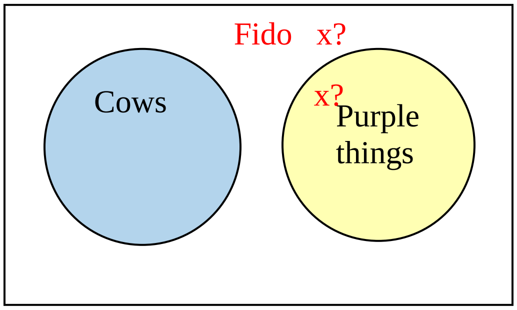
9. Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.



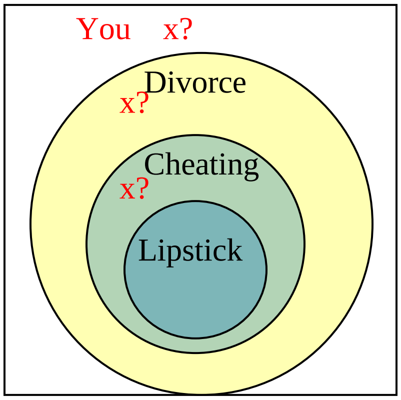
10. Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.



11. Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.



12. Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don't have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren't cheating on me, I might divorce you for another reason. You'd better shape up.



13. Let S = have a shovel, D = dig a hole.
The first premise is equivalent to $S \to D$. The second premise is D. The conclusion is S.
We are testing $[(S \to D) \wedge D] \to S$

S	D	$S \to D$	$(S \to D) \wedge D$	$[(S \to D) \wedge D] \to S$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

This is not a tautology, so this is an invalid argument.

14. Let b = brushed teeth and w = toothbrush is wet.
Premise: $b \rightarrow w$
Premise: $\sim w$
Conclusion: $\sim b$
This argument is valid by the Law of Contraposition.

15. This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \rightarrow s$, $s \rightarrow b$, $b \rightarrow c$, $c \rightarrow d$, $d \rightarrow g$, $g \rightarrow w$, $w \rightarrow h$, $h \rightarrow x$ can be reduced to $f \rightarrow x$. (Because we had already used c and d, we decided to use w for cow and x for death.) If the old lady swallows the fly, she will eventually eat a horse and die.

16. Let p = wrote a paper and s = gave a speech.
Premise: $p \vee s$
Premise: $\sim s$
Conclusion: p
This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

17. Let f = pulled fire alarm and t = got in big trouble.
Premise: $f \rightarrow t$
Premise: t
Conclusion: f

This argument is invalid because it has the form of the Fallacy of the Converse. The young rascal may have gotten in trouble for any number of reasons besides pulling the fire alarm.

18. Let t = tripped and p = got a penalty.
Premise: $t \rightarrow p$
Premise: $\sim t$
Conclusion: $\sim p$

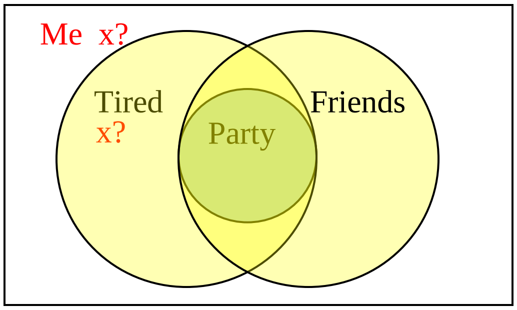
This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

19. Let p = go to party, t = be tired, and f = see friends.
Premise: $p \rightarrow t$
Premise: $p \rightarrow f$
Conclusion: $\sim f \rightarrow \sim t$

We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.

(Explanation continued on next page)

A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.



20.a. Circular
b. Correlation does not imply causation
c. Post hoc
d. Appeal to consequence
e. Straw man

Exercises

Boolean Logic

For questions 1-2, list the set of integers that satisfy the given conditions.

1. A positive multiple of 5 and not a multiple of 2

2. Greater than 12 and less than or equal to 18

Quantified Statements

For questions 3-4, write the negation of each quantified statement.

3. Everyone failed the quiz today.

4. Someone in the car needs to use the restroom.

Truth Tables

5. Translate each statement from symbolic notation into English sentences. Let A represent "Elvis is alive" and let G represent "Elvis gained weight".
   a. $A \lor G$
   b. $\sim(A \land G)$
   c. $G \rightarrow \sim A$
   d. $A \leftrightarrow \sim G$

For questions 6-9, create a truth table for each statement.

6. $A \land \sim B$

7. $\sim(\sim A \lor B)$

8. $(A \land B) \rightarrow C$

9. $(A \lor B) \rightarrow \sim C$

Questions 10-13: In this lesson, we have been studying the *inclusive* or, which allows both A and B to be true. The *exclusive* or does not allow both to be true; it translates to "either A or B, but not both."

10. For each situation, decide whether the "or" is most likely exclusive or inclusive.
   a. An entrée at a restaurant includes soup or a salad.
   b. You should bring an umbrella or a raincoat with you.
   c. We can keep driving on I-5 or get on I-405 at the next exit.
   d. You should save this document on your computer or a flash drive.

11. Complete the truth table for the exclusive or.

A	B	$A \veebar B$
T	T	
T	F	
F	T	
F	F	

12. Complete the truth table for $(A \vee B) \wedge \sim(A \wedge B)$.

A	B	$A \vee B$	$A \wedge B$	$\sim(A \wedge B)$	$(A \vee B) \wedge \sim(A \wedge B)$
T	T				
T	F				
F	T				
F	F				

13. Compare your answers for questions 11 and 12. Can you explain the similarities?

Conditional Statements

14. Consider the statement "If you are under age 17, then you cannot attend this movie."
    a. Write the converse.
    b. Write the inverse.
    c. Write the contrapositive.

15. Assume that the statement "If you swear, then you will get your mouth washed out with soap" is true. Which of the following statements must also be true?
    a. If you don't swear, then you won't get your mouth washed out with soap.
    b. If you don't get your mouth washed out with soap, then you didn't swear.
    c. If you get your mouth washed out with soap, then you swore.

For questions 16-18, write the negation of each conditional statement.

16. If you don't look both ways before crossing the street, then you will get hit by a car.

17. If Luke faces Vader, then Obi-Wan cannot interfere.

18. If you weren't talking, then you wouldn't have missed the instructions.

19. Assume that the biconditional statement "You will play in the game if and only if you attend all practices this week" is true. Which of the following situations could happen?
    a. You attended all practices this week and didn't play in the game.
    b. You didn't attend all practices this week and played in the game.
    c. You didn't attend all practices this week and didn't play in the game.

De Morgan's Laws

For questions 20-21, use De Morgan's Laws to rewrite each conjunction as a disjunction, or each disjunction as a conjunction.

20. It is not true that Tina likes Sprite or 7-Up.

21. It is not the case that you need a dated receipt and your credit card to return this item.

22. Go back and look at the truth tables in Exercises 6 & 7. Explain why the results are identical.

Deductive Arguments

For questions 23-28, use a Venn diagram or truth table or common form of an argument to decide whether each argument is valid or invalid.

23. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.

24. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.

25. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose down the garbage cans.

26. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!

27. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

28. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

Logical Fallacies

For questions 28-30, name the type of logical fallacy being used.

29. If you don't want to drive from Boston to New York, then you will have to take the train.

30. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

31. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.

CHAPTER 6
FRACTALS

Excerpted from *Math in Society*, by David Lippman

Fractals

Fractals are geometric figures, usually obtained through recursion, such that parts of the figure look like scaled versions of the whole. More precisely, fractals can be defined as figures that are self-similar and that exhibit certain dimensional properties. Our main focus will be on the concept of self-similarity, and the recursive procedures used to generate fractals.

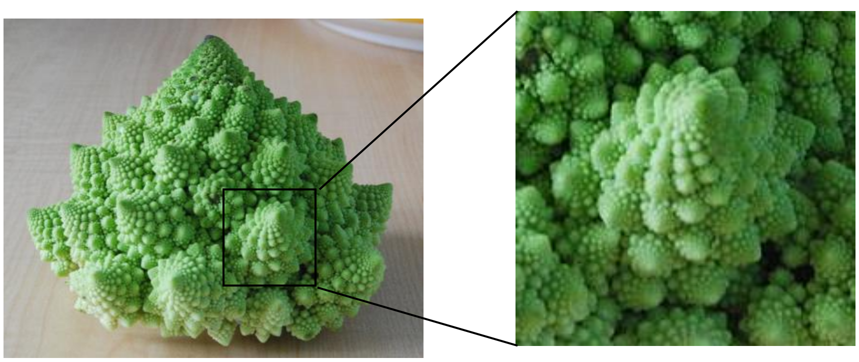
Sec. 6.1 Basic Definitions

One of the most important features of fractals is that they appear the same at different scales. This property is known as self-similarity.

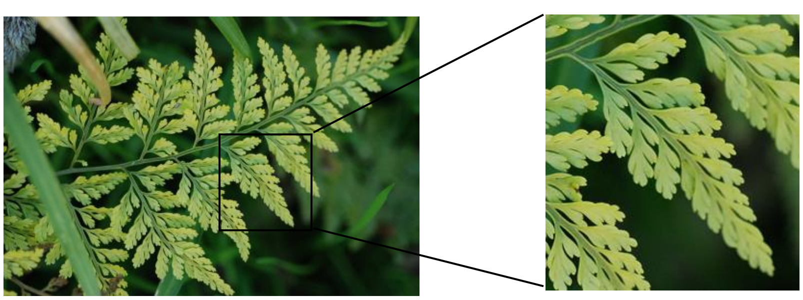
Self-similarity
A geometric figure is **self-similar** if it is exactly or approximately **similar** to one or more parts of itself.

Intuitively, this means that the figure looks essentially the same from a distance as it does closer up.

Self-similarity often appears in nature. In the Romanesco broccoli pictured below[1], if we zoom in on part of the image, the enlarged portion looks similar to the whole.

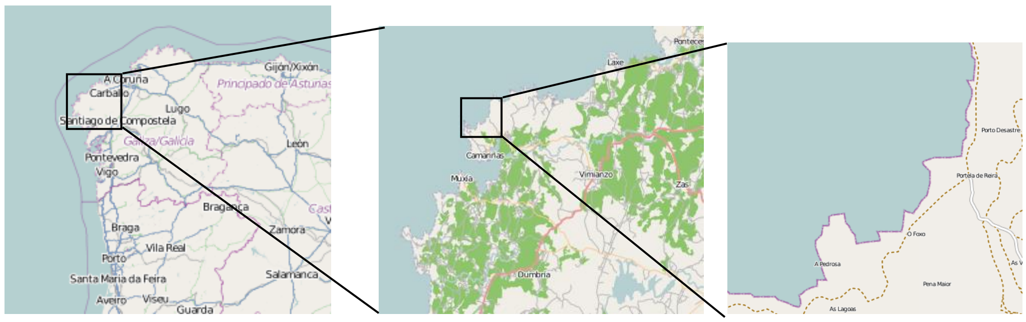


Likewise, in the fern frond below[2], one piece of the frond looks similar to the whole.



[1] http://en.wikipedia.org/wiki/File:Cauliflower_Fractal_AVM.JPG
[2] http://www.flickr.com/photos/cjewel/3261398909/

This phenomenon also shows up in the physical sciences. For example, if we zoom in on the coastline of Portugal[3], each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.



Iterated Fractals

This self-similar behavior can be replicated through recursion: repeating a process over and over. That is, we can construct fractals by starting with a simple figure, and then repeatedly altering it according to a specified rule. To formalize this process we next introduce the concept of initiators and generators:

Initiators and Generators

An **initiator** is a starting shape.
A **generator** is an arranged collection of scaled copies of the initiator

To generate fractals from initiators and generators, we follow a simple rule:

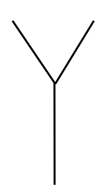
Fractal Generation Rule
At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary

This process is easiest to understand through examples.

Example 1

Use the initiator and generator below, however only iterate on the "branches." Sketch several steps of the iteration.

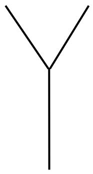
initiator



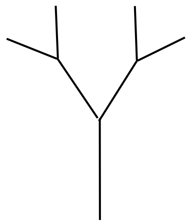
generator

[3] Openstreetmap.org, CC-BY-SA

We begin by replacing the initiator with the generator. We then replace each "branch" of Step 1 with a scaled copy of the generator to create Step 2.

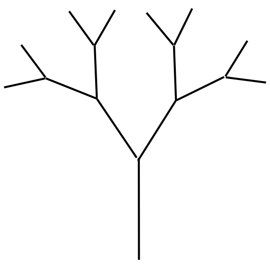


Step 1

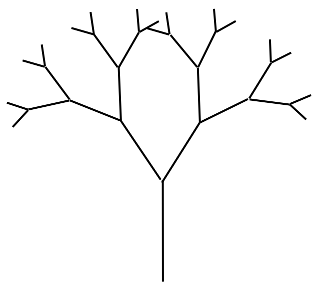


Step 2

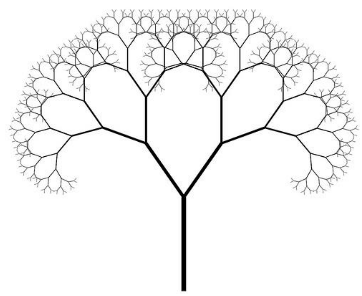
We can repeat this process to create later steps.   Repeating this process can create intricate tree shapes[4].



Step 3

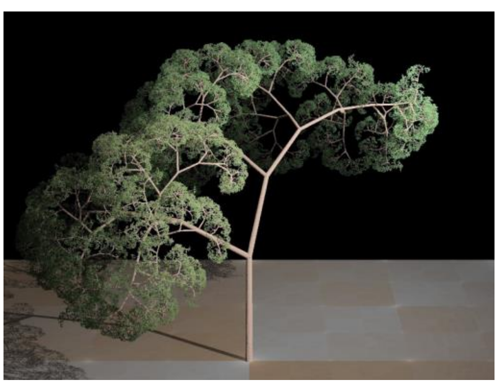


Step 4



Final shape

Using iteration processes like those above can create a variety of beautiful images evocative of nature[5,6].



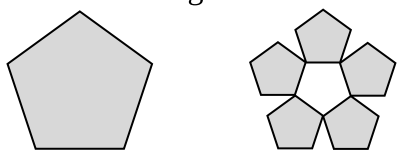
[4] http://www.flickr.com/photos/visualarts/5436068969/

[5] http://en.wikipedia.org/wiki/File:Fractal_tree_%28Plate_b_-_2%29.jpg

[6] http://en.wikipedia.org/wiki/File:Barnsley_Fern_fractals_-_4_states.PNG

Try it Now 1

Use the initiator and generator shown to produce the next two stages



Initiator Generator

Sec. 6.2 The Koch Snowflake

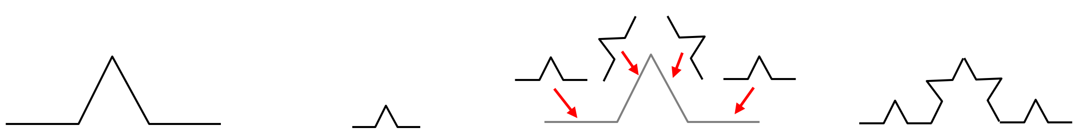
One of the earliest and most famous fractals to be described using the iterative process in the previous section is the Koch snowflake (pronounced "coke"). This figure is named for the Swedish mathematician Helge von Koch who first studied it in 1904. We'll start by describing the Koch curve, on which the snowflake is based.

Example 2 – The Koch curve

Use the initiator and generator shown to create the iterated fractal.

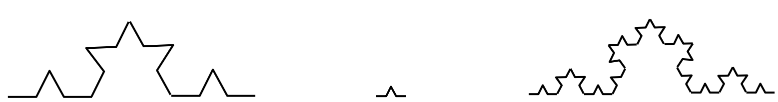
initiator generator

At each step, we replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four different line segments of step 1 is replaced with a scaled copy of the generator:



Step 1 | Scaled copy of generator | A scaled copy replaces each line segment of Step 1 | Step 2

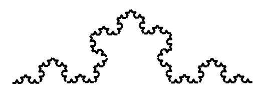
This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.



Step 2 | Scaled copy of generator | Step 3

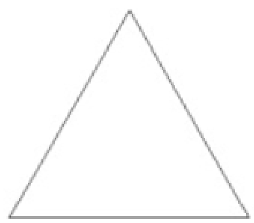
Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0.
Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator.
Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator.
Step 4, then, would require 16*4 = 64 copies of the generator.

The shape resulting from iterating this process indefinitely is called the Koch curve:

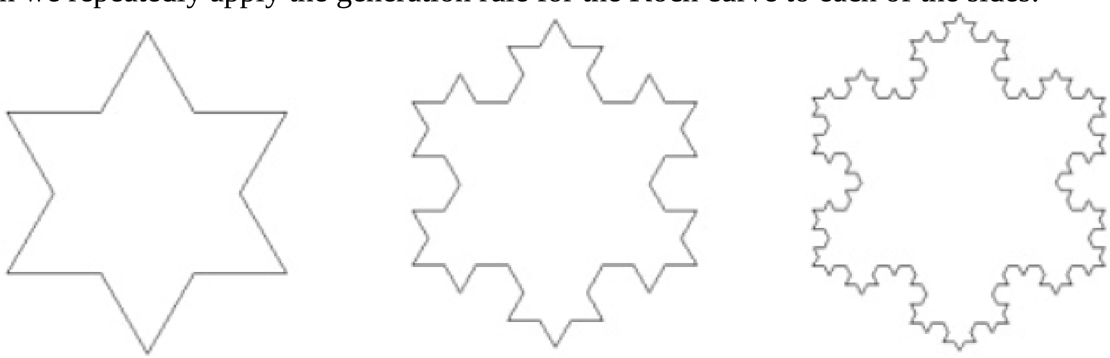


Koch curve

Now that we understand the Koch curve, we can easily describe the Koch snowflake.
We start with an equilateral triangle:

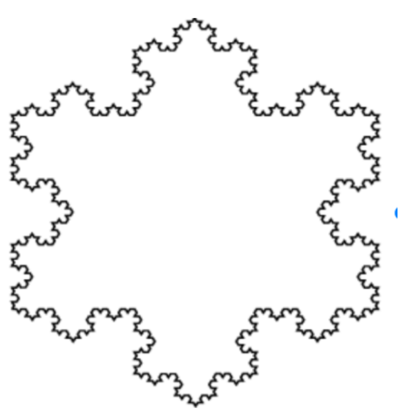


Then we repeatedly apply the generation rule for the Koch curve to each of the sides:



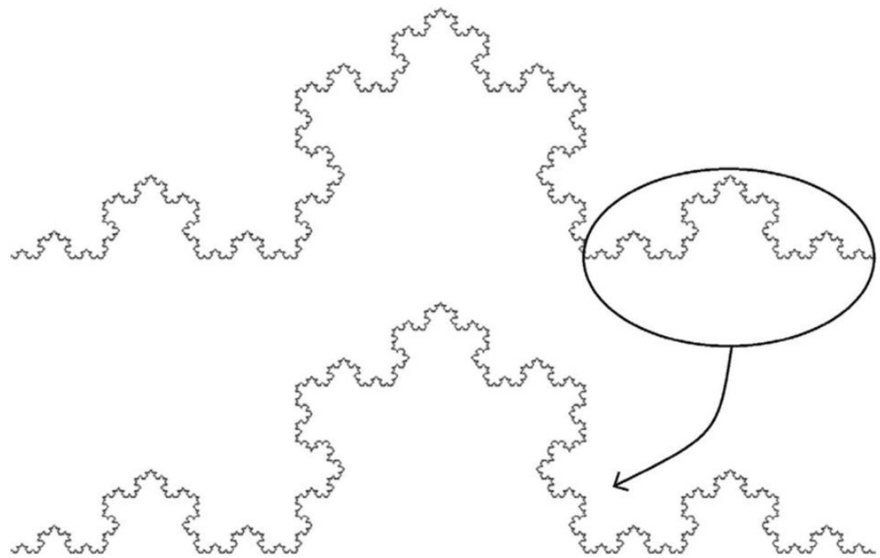
Step 1 Step 2 Step 3

Note that even after just three iterations, the figure does resemble a snowflake. Repeating this process indefinitely (i.e. without end) to obtain the Koch snowflake:

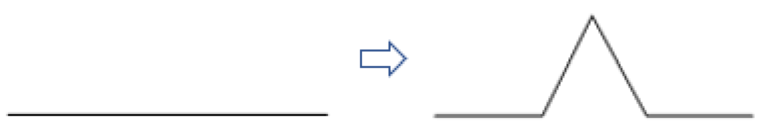


Of course, we cannot *really* repeat the process infinitely many times, but even so we can deduce some interesting properties of the completed Koch snowflake.

First, if we zoom in on any section of the snowflake, we will see a Koch curve. And the Koch curve is perfectly self-similar:



Second, it is clear from the generation rule



that the perimeter of the snowflake increases at each step of the construction. Moreover, we can describe precisely how it changes. At each step, we replace a single line segment with four smaller line segments, each has 1/3 the length of the original segment. Thus, at each step of the construction, the perimeter will increase by a factor or 4/3. To better understand this, let's let P_n represent the perimeter of the snowflake at step n, where $n = 0$ corresponds to the starting triangle. Then the remarks above tell us that the perimeter at any given stage is given by:

$$P_{n+1} = \frac{4}{3} P_n$$

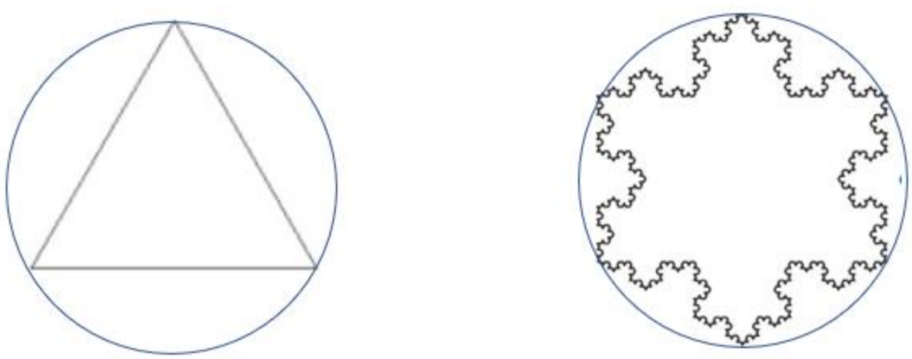
This says that the perimeter at any step about 1.33 times the perimeter at the previous step. To get an idea of just how fast the perimeter is growing, let's assume that the starting triangle where each side has 1 inch, so the starting perimeter is 3 inches.

Now, make a table:

n	P_n (in inches)
0	3
1	4
2	9.33
3	12.44
4	16.59
:	
100	9.35×10^{12}

We see that after 100 steps, the perimeter of the partially completed snowflake is about 9.35 *trillion* inches. This is more than 1½ times the distance from the earth to the sun! And the perimeter will continue to grow without bound – in other words, the perimeter of the completed Koch snowflake is **infinite**.

On the other hand, the area enclosed by the Koch snowflake is finite. We can see this because if we circumscribe a circle around the starting triangle, the same circle will circumscribe the completed snowflake:



Since the snowflake fits into a circle of finite area, the area enclosed by the snowflake must also be finite. Using techniques from Calculus, we can show that the area enclosed by the completed Koch snowflake is exactly 1.6 times the area of the starting triangle.

Thus, the Koch snowflake is a curve of infinite length that encloses a region of finite area. In summary,

Properties of the Koch Snowflake

1. If P_n is the perimeter of the figure at the n^{th} step, then the perimeter of the figure at the next step is $P_{n+1} = (4/3)P_n$.
2. Because the perimeter increases without bound, the perimeter of the completed Koch snowflake is **infinite.**
3. The area of the completed Koch snowflake is $A = 1.6A_0$, where A_0 is the area of the starting triangle.

Try it Now 2

Suppose that the starting equilateral triangle of a Koch snowflake has perimeter 6 inches and area 3.46 square inches.

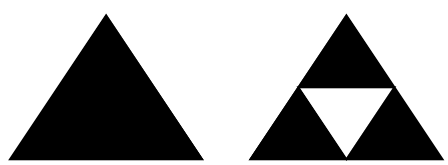
a) What is the perimeter of the figure after the first step of the construction?
b) What is the perimeter of the figure after the fourth step of the construction?
c) What is the area of the completed Koch snowflake?

Sec. 6.3 The Sierpinski Gasket

Another well-known and extensively studied fractal is the Sierpinski gasket, named after the Polish mathematician Waclaw Sierpinski, who published his landmark paper on the subject in 1915. We describe the iterative process for generating this figure as follows:

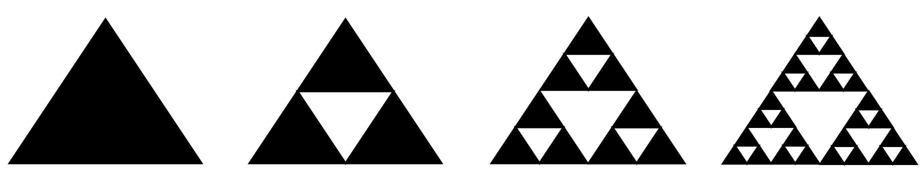
Example 3

Start with a filled-in equilateral triangle. We connect the midpoints of each side and remove the middle triangle. In other words, the initiator and generator for this construction are:



Initiator Generator

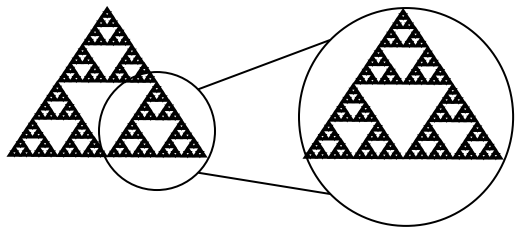
We then repeat this process; here are the first three iterations:



Initial Step 1 Step 2 Step 3

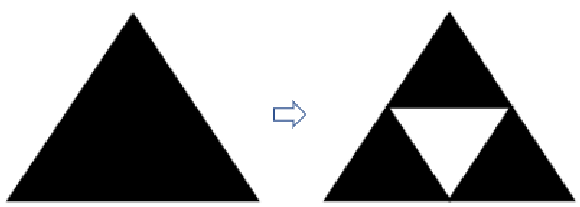
Repeating this process indefinitely, the shape that emerges is called the Sierpinski gasket. Observe that it exhibits self-similarity – any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original.

Of course, each of those copies also contains three copies of itself:



Thus, the Sierpinski gasket is perfectly self-similar.

We can deduce some other basic properties of the Sierpinski gasket using the generation rule:



First, we can see that the shaded area will decrease with every iteration.   More precisely, at each step we remove one of four congruent triangles; thus at each step we remove ¼ of the shaded area. So at each step, the shaded area is ¾ of the shaded area in the previous step. Again, this can be stated more precisely. If we let A_n represent the shaded area of the figure at step n (where $n = 0$ corresponds to the starting triangle), then we have the following:

$$A_{n+1} = \frac{3}{4}A_n$$

To better understand how the area is decreasing, let's assume that the starting triangle has area 4 square inches, and make a table:

n	A_n (in square inches)
0	4
1	3
2	2.25
3	1.6875
4	1.2656
:	
100	0.00000000000128

We see that after 100 steps, the shaded area is about 13 *trillionths* of a square inch. And this area will continue to decrease, getting ever closer to zero. In other words, the completed Sierpinski gasket has **zero area**.

Second, we see that the total perimeter of the Sierpinski gasket will increase with every iteration. Here we define the total perimeter to be the total of the boundaries of all shaded regions. Again referring to the generation rule, we see that at each step, we will have the perimeter from the previous step, and then we add the perimeter of the "whited out" interior triangle. This smaller triangle is a scaled version of the large triangle, with scaling factor ½, so the perimeter of the small triangle is ½ the perimeter of the large one. Thus the total perimeter in any given step is 1½ the total perimeter from the previous step. Again we let P_n represent the total perimeter of the figure at step n, where $n = 0$ corresponds to the starting triangle. Then we can concisely summarize the analysis above concisely:

$$P_{n+1} = 1.5\, P_n$$

Again we can see how fast the total perimeter is growing by an example. Let the starting triangle have perimeter 3 inches, and make a table:

n	P_n (in inches)
0	3
1	4.5
2	6.75
3	10.125
4	15.1875
:	
100	1.22×10^{18}

Thus, after the 100th step, the perimeter of the gasket is about 2.22 *quadrillion* inches. Again, this is an astronomical number (literally), about two light years.

Thus, the Sierpinski gasket is a curve of infinite length that encloses a region with zero area. We summarize our findings so far:

Properties of the Sierpinski Gasket

1. If P_n is the total perimeter of the figure at the n^{th} step, then the perimeter of the figure at the next step is $P_{n+1} = 1.5P_n$.
2. Because the perimeter increases without bound, the total perimeter of the completed Sierpinski gasket is **infinite.**
3. If A_n is the total shaded area of the figure at the n^{th} step, then the total shaded area at the next step is $A_{n+1} = ¾\, A_n$
4. Because the area decreases and gets arbitrarily close to zero, the total shaded area of the completed Sierpinski gasket is **zero**.

Try it Now 3

Suppose that the starting equilateral triangle for a Sierpinski gasket has perimeter 6 inches and area 3.46 square inches.

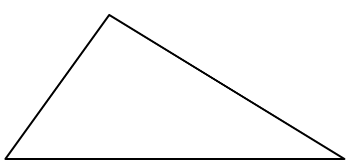
a) What is the perimeter of the figure after the first step of the construction?
b) What is the perimeter of the figure after the third step of the construction?
c) What is the perimeter of the completed Sierpinski gasket?
d) What is the area of the figure after the first step of the construction?
e) What is the area of the figure after the third step of the construction?
f) What is the area of the completed Sierpinski gasket?

The Sierpinski gasket has some interesting applications. For example, a partially constructed gasket is used in the design of antennas for cell phone towers (called fractal antennas). Because the figure has very little area, it has "low profile" (less susceptible to damage from the elements) and the large perimeter allows for large bandwidth. Moreover, a variation on the Sierpinski gasket can be used in computer modeling of natural phenomena.

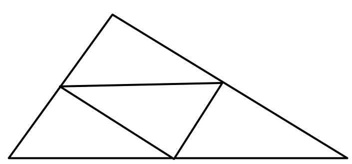
Example 4

Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made.

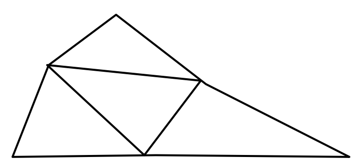
Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.



Step 0

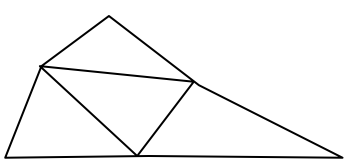


Step 1

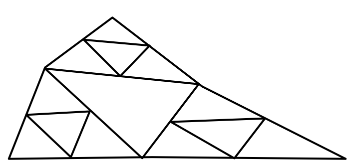


Step 1 with randomnesss

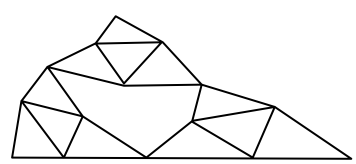
We then repeat this process.



Step 1 with randomnesss



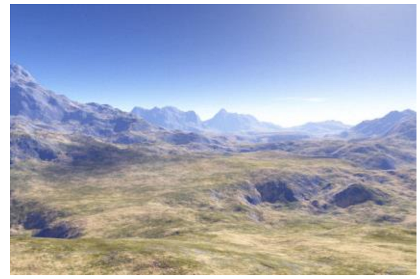
Step 2



Step 2 with randomness

Continuing this process can create mountain-like structures.

The landscape to the right[7] was created using fractals, then colored and textured.

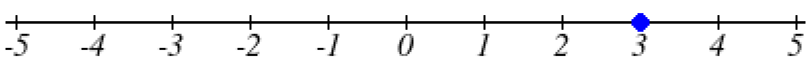


Sec. 6.4: The Mandelbrot Set

We will now turn our attention to another type of fractal, defined by a different type of recursion. To understand this type, we first need to review some basic properties of complex numbers.

Complex Numbers[8]

The numbers you are most familiar with are called **real numbers**. These include numbers like 4, 275, -200, 10.7, ½, π, and so forth. All these real numbers can be plotted on a number line. For example, if we wanted to show the number 3, we plot a point:



To solve certain problems like $x^2 = -4$, it became necessary to introduce **imaginary numbers**.

Imaginary Number *i*
The imaginary number i is defined to be $i = \sqrt{-1}$.
Any real multiple of i (e.g. $5i$) is also an imaginary number.

Example 5

Simplify $\sqrt{-9}$.

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$.
We can take the square root of 9, and write the square root of -1 as i to get

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

[7] http://en.wikipedia.org/wiki/File:FractalLandscape.jpg
[8] Portions of this section are remixed from Precalculus: An Investigation of Functions by David Lippman and Melonie Rasmussen. CC-BY-SA

A complex number is the sum of a real number and an imaginary number.

Complex Number

A **complex number** is a number $z = a + bi$, where a and b are real numbers

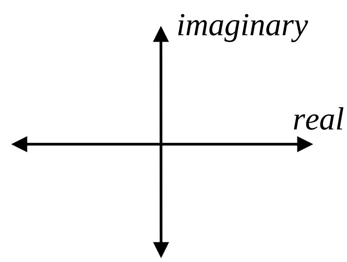
a is called the *real part* of the complex number

b is called the *imaginary part* of the complex number

To plot a complex number like $3 - 4i$, we need more than just a number line since there are two components to the number. To plot this number, we need two number lines, crossed to form a complex plane.

Complex Plane

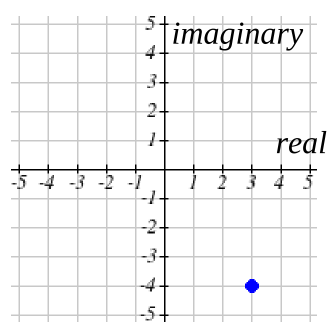
In the **complex plane**, the horizontal axis is the real axis and the vertical axis is the imaginary axis.



Example 6

Plot the number $3 - 4i$ on the complex plane.

The real part of this number is 3, and the imaginary part is -4. To plot this, we draw a point 3 units to the right of the origin in the horizontal direction and 4 units down in the vertical direction.



Because this is analogous to the Cartesian coordinate system for plotting points, we can think about plotting our complex number $z = a + bi$ as if we were plotting the point (a, b) in Cartesian coordinates. Sometimes people write complex numbers as $z = x + yi$ to highlight this relation.

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example 7

Add $3-4i$ and $2+5i$.

Adding $(3-4i)+(2+5i)$, we add the real parts and the imaginary parts and then simplify by combining like terms:

$(3 - 4i) + (2 + 5i) = 3 + 2 - 4i + 5i = 5 + i.$

Try it Now 4

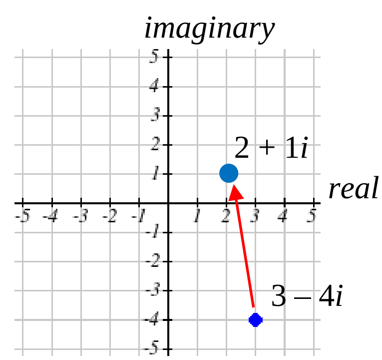
Subtract $2+5i$ from $3-4i$

When we add complex numbers, we can visualize the addition as a shift, or translation, of a point in the complex plane.

Example 8

Visualize the addition $3-4i$ and $-1 + 5i$.

The initial point is $3-4i$. When we add $-1 + 5i$. we add -1 to the real part, moving the point 1 unit to the left, and we add 5 to the imaginary part, moving the point 5 units vertically. This shifts the point $3-4i$ to $2+1i$



We can also multiply complex numbers by a real number, or multiply two complex numbers.

Example 9

Multiply: $4(2+5i)$.

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials:

$$\begin{aligned} 4(2+5i) &= 4\cdot 2+4\cdot 5i && \text{Distribute} \\ &= 8+20i && \text{Simplify} \end{aligned}$$

Example 10

Multiply: $(2+5i)(4+i)$.

$(2+5i)(4+i)$	$=8+20i+2i+5i^2$	Expand
	$=8+20i+2i+5(-1)$	Since $i=\sqrt{-1}$, $i^2=-1$
	$=3+22i$	Simplify

Try it Now 5

Multiply $3-4i$ and $2+3i$.

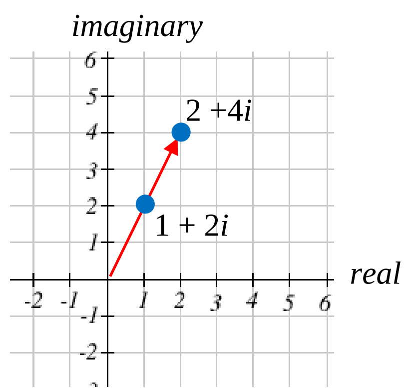
To understand the effect of multiplication visually, we'll explore three examples.

Example 11

Visualize the product $2(1+2i)$

Multiplying we get

$2\cdot 1 + 2\cdot 2i = 2 + 4i$



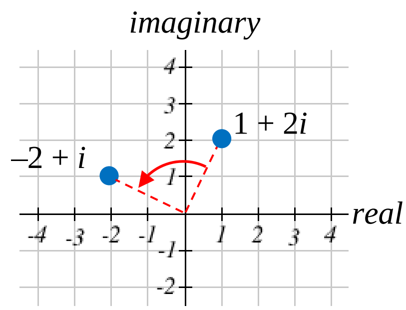
Notice both the real and imaginary parts have been scaled by 2. Visually, this will stretch the point outwards, away from the origin.

Example 12

Visualize the product $i(1+2i)$

Multiplying, we get

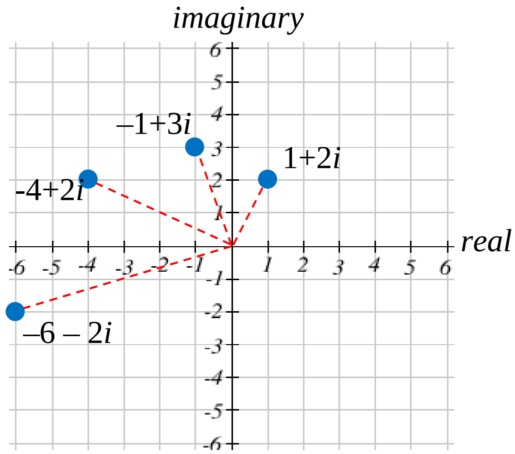
$$\begin{aligned} i\cdot 1 + i\cdot 2i &= i\cdot 1 + 2i^2 \\ &= i + 2(-1) \\ &= -2 + i \end{aligned}$$



In this case, the distance from the origin has not changed, but the point has been rotated about the origin, 90° counter-clockwise.

Example 13

Visualize the result of multiplying $1+2i$ by $1+i$. Then show the result of multiplying by $1+i$ again.



Multiplying $1+2i$ by $1+i$,

$$(1+2i)(1+i)$$
$$=1+i+2i+2i^2$$
$$=1+3i+2(-1)$$
$$=-1+3i$$

Multiplying by $1+i$ again,

$$(-1+3i)(1+i)$$
$$=-1-i+3i+3i^2$$
$$=-1+2i+3(-1)$$
$$=-4+2i$$

If we multiplied by $1+i$ again, we'd get $-6-2i$. Plotting these numbers in the complex plane, you may notice that each point gets both further from the origin, and rotates counterclockwise, in this case by 45°.

In general, multiplication by a complex number can be thought of as a **scaling**, changing the distance from the origin, combined with a **rotation** about the origin.

Complex Recursive Sequences

We will now explore recursively defined sequences of complex numbers.

Recursive Sequence

A **recursive relationship** is a formula which relates the next value, z_{n+1}, in a sequence to the previous value, z_n. In addition to the formula, we need an initial value, z_0.

The sequence of values produced is the recursive sequence.

Example 14

Given the recursive relationship $z_{n+1}=z_n+2, \quad z_0=4$, generate several terms of the recursive sequence.

We are given the starting value, $z_0=4$. The recursive formula holds for any value of *n*, so if *n* = 0, then $z_{n+1}=z_n+2$ would tell us $z_{0+1}=z_0+2$, or more simply, $z_1=z_0+2$.

Notice this defines z_1 in terms of the known z_0, so we can compute the value:

$$z_1 = z_0 + 2 = 4 + 2 = 6.$$

Now letting n = 1, the formula tells us $z_{1+1} = z_1 + 2$, or $z_2 = z_1 + 2$.
Again, the formula gives the next value in the sequence in terms of the previous value:

$$z_2 = z_1 + 2 = 6 + 2 = 8$$

Continuing,

$$z_3 = z_2 + 2 = 8 + 2 = 10$$

$$z_4 = z_3 + 2 = 10 + 2 = 12$$

The previous example generated a basic linear sequence of real numbers. The same process can be used with complex numbers. (Note that we also used recursive sequences to analyze the perimeter and area properties of the Koch snowflake and Sierpinski gasket)

Example 15

Given the recursive relationship $z_{n+1} = z_n \cdot i + (1-i), \quad z_0 = 4$, generate several terms of the recursive sequence.

We are given $z_0 = 4$. Using the recursive formula:

$$z_1 = z_0 \cdot i + (1-i) = 4 \cdot i + (1-i) = 1 + 3i$$
$$z_2 = z_1 \cdot i + (1-i) = (1+3i) \cdot i + (1-i) = i + 3i^2 + (1-i) = i - 3 + (1-i) = -2$$
$$z_3 = z_2 \cdot i + (1-i) = (-2) \cdot i + (1-i) = -2i + (1-i) = 1 - 3i$$
$$z_4 = z_3 \cdot i + (1-i) = (1-3i) \cdot i + (1-i) = i - 3i^2 + (1-i) = i + 3 + (1-i) = 4$$
$$z_5 = z_4 \cdot i + (1-i) = 4 \cdot i + (1-i) = 1 + 3i$$

Notice this sequence is exhibiting an interesting pattern – it began to repeat itself.

Mandelbrot Set

The Mandelbrot Set is a set of numbers defined based on recursive sequences

Mandelbrot Set

For any complex number c, define the sequence $z_{n+1} = z_n^{\,2} + c, \quad z_0 = 0$

If this sequence always stays close to the origin (within 2 units), then the number c is part of the **Mandelbrot Set**. If the sequence gets far from the origin, then the number c is not part of the set.

Example 16

Determine if $c = 1 + i$ is part of the Mandelbrot set.

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$z_1 = z_0{}^2 + 1 + i = 0 + 1 + i = 1 + i$

$z_2 = z_1{}^2 + 1 + i = (1+i)^2 + 1 + i = 1 + 3i$

$z_3 = z_2{}^2 + 1 + i = (1+3i)^2 + 1 + i = -7 + 7i$

$z_4 = z_3{}^2 + 1 + i = (-7+7i)^2 + 1 + i = 1 - 97i$

We can already see that these values are getting quite large. It does not appear that $c = 1 + i$ is part of the Mandelbrot set.

Example 17

Determine if $c = 0.5i$ is part of the Mandelbrot set.

We start with $z_0 = 0$. We continue, omitting some detail of the calculations

$z_1 = z_0{}^2 + 0.5i = 0 + 0.5i = 0.5i$

$z_2 = z_1{}^2 + 0.5i = (0.5i)^2 + 0.5i = -0.25 + 0.5i$

$z_3 = z_2{}^2 + 0.5i = (-0.25 + 0.5i)^2 + 0.5i = -0.1875 + 0.25i$

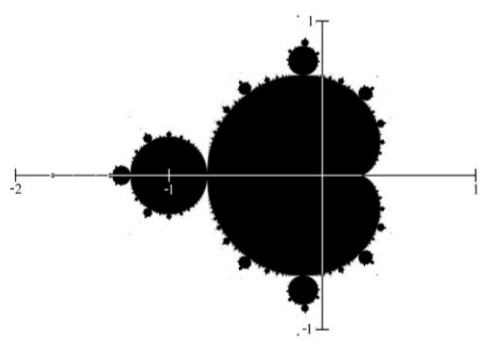
$z_4 = z_3{}^2 + 0.5i = (-0.1875 + 0.25i)^2 + 0.5i = -0.02734 + 0.40625\,i$

While not definitive with this few iterations, it does appear that this value is remaining small, suggesting that $0.5i$ is part of the Mandelbrot set.

Try it Now 6

Determine if $c = 0.4 + 0.3i$ is part of the Mandelbrot set.

If all complex numbers are tested, and we plot each number that is in the Mandelbrot set on the complex plane, we obtain the shape to the right[9].



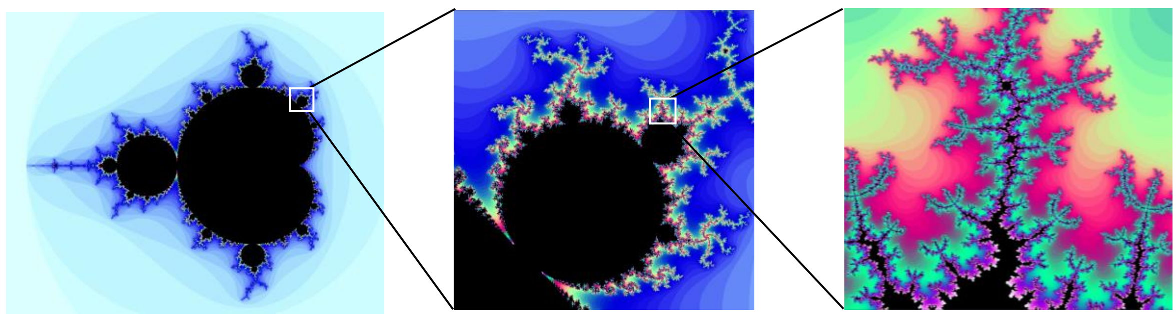
The boundary of this shape exhibits quasi-self-similarity, in that portions look very similar to the whole.

In addition to coloring the Mandelbrot set itself black, it is common to the color the points in the complex plane surrounding the set. To create a meaningful coloring, often people count the

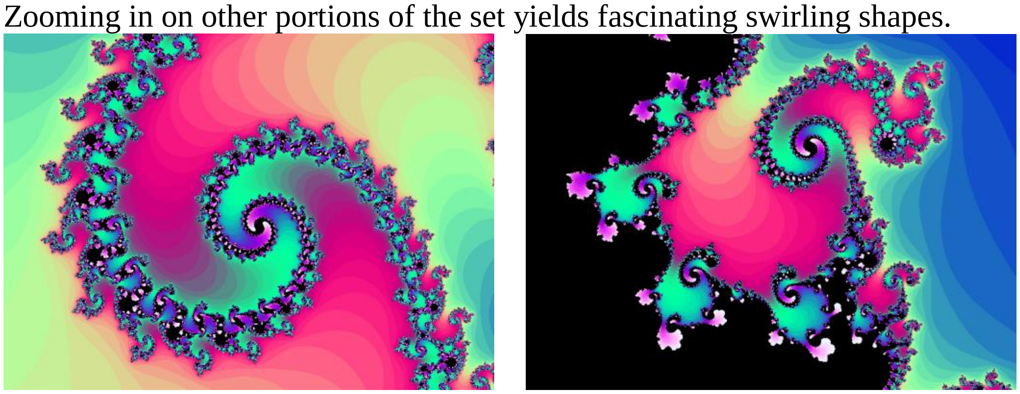
[9] http://en.wikipedia.org/wiki/File:Mandelset_hires.png

number of iterations of the recursive sequence that are required for a point to get further than 2 units away from the origin. For example, using $c = 1 + i$ above, the sequence was distance 2 from the origin after only two recursions.

For some other numbers, it may take tens or hundreds of iterations for the sequence to get far from the origin. Numbers that get big fast are colored one shade, while colors that are slow to grow are colored another shade. For example, in the image below[10], light blue is used for numbers that get large quickly, while darker shades are used for numbers that grow more slowly. Greens, reds, and purples can be seen when we zoom in – those are used for numbers that grow very slowly.



The Mandelbrot set, for having such a simple definition, exhibits immense complexity. Zooming in on other portions of the set yields fascinating swirling shapes.

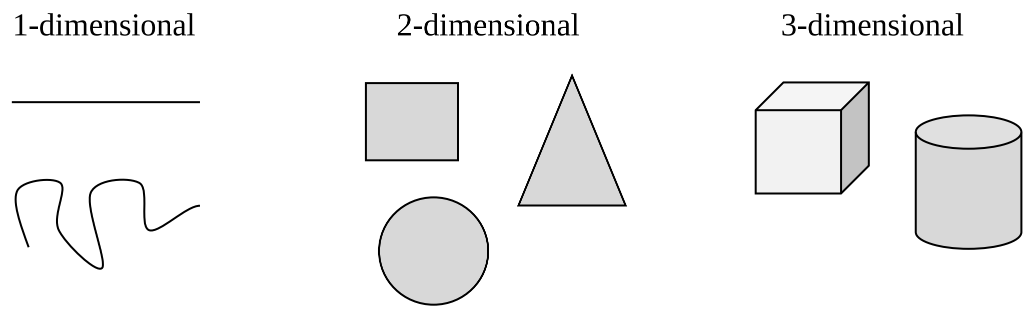


Sec. 6.5: Fractal Dimension

In addition to visual self-similarity, fractals exhibit other interesting properties. For example, recall that in each step of the Sierpinski gasket, we removed one quarter of the remaining area. Subsequently, we saw that when the process is continued indefinitely, we end up essentially removing all the area. This means that we started with a 2-dimensional area, and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

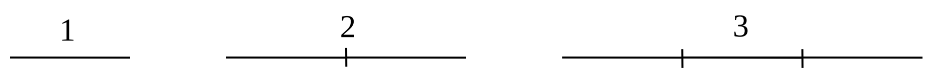
[10] This series was generated using [Scott's Mandelbrot Set Explorer]

To explore this idea, we need to discuss dimension. A geometric figure like a line is called 1-dimensional, since it only has length. Likewise, any curve is 1-dimensional. Figures like boxes and circles are 2-dimensional, since they have length and width, and enclose an area. Objects like boxes and cylinders have length, width, and height, enclosing a volume, and are called 3-dimensional.

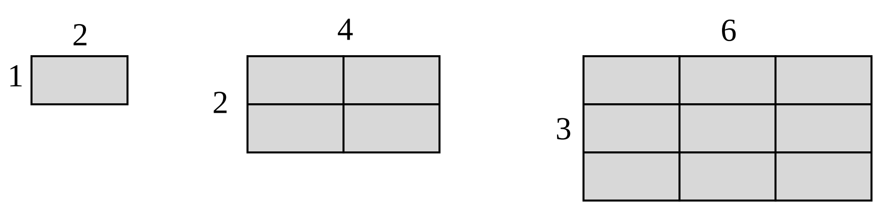


Certain rules apply for scaling objects, related to their dimension.

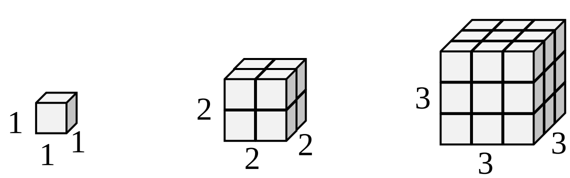
If we had a line with length 1, and wanted scale its length by 2, we would need two copies of the original line. If we had a line of length 1, and wanted to scale its length by 3, we would need three copies of the original:



If we had a rectangle with length 2 and height 1, and wanted to scale its length and width by 2, we would need four copies of the original rectangle. If we wanted to scale the length and width by 3, we would need nine copies of the original rectangle:



If we had a cubical box with sides of length 1, and wanted to scale its length and width by 2, we would need eight copies of the original cube. If we wanted to scale the length and width by 3, we would need 27 copies of the original cube.



Notice that:

- In the 1-dimensional case, number of copies needed = scale.
- In the 2-dimensional case, number of copies needed = scale^2.
- In the 3-dimensional case, number of copies needed = scale^3.

From these examples, we might infer a pattern.

Scaling-Dimension Relation

To scale a D-dimensional shape by a scaling factor S, the number of copies C of the original shape needed will be given by:

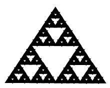
Copies = Scale$^{\text{Dimension}}$, or $C = S^D$

Example 18

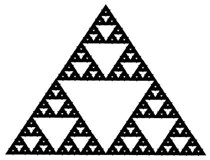
Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2.

Notice that to construct the larger gasket, 3 copies of the original gasket are needed.



1



2

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$.

Since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1-dimensional shape, but we've taken away so much area that it is now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula):

$3 = 2^D$

$\log(3) = \log\left(2^D\right)$ Take the logarithm of both sides

$\log(3) = D\log(2)$ Use the exponent property of logs

$D = \dfrac{\log(3)}{\log(2)} \approx 1.585$ Divide by log(2)

Thus, the dimension of the gasket is about 1.585

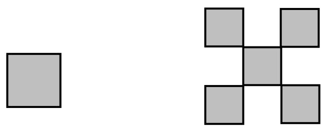
Calculating the Dimension of a Fractal

To find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula:

$$D = \frac{\log(C)}{\log(S)}$$

Try it Now 7

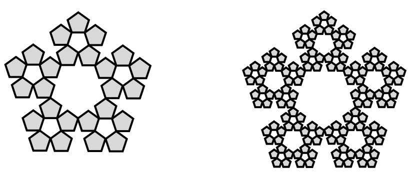
Determine the fractal dimension of the fractal produced using the initiator and generator



Initiator Generator

Try it Now Answers

1.



Step 2 Step 3

2. Let P_n be the perimeter of the figure at the n^{th} step, so the perimeter of the figure at the next step is $P_{n+1} = (4/3)P_n$.

a) $P_1 = (4/3)P_0 = (4/3) \times 6 = 8$ inches.
b) By iteration, $P_4 = (4/3)^4 P_0 = (4/3)^4 \times 6 = 18.96$ inches
c) The area is $A = 1.6 \times 3.46 = 5.536$ square inches

3. Let P_n be the perimeter of the figure at the n^{th} step, so the perimeter of the figure at the next step is $P_{n+1} = (1.5)P_n$.

a) $P_1 = 1.5\ P_0 = 1.5 \times 6 = 9$ inches.
b) By iteration, $P_3 = 1.5^3 P_0 = 1.5^3 \times 6 = 20.25$ inches
c) The total perimeter of the completed figure is **infinite**.

Let A_n be the area of the shaded portions of the figure at the n^{th} step, so the area of the figure at the next step is $A_{n+1} = ¾ A_n$.

d) $A_1 = ¾ A_0 = ¾ \times 3.46 \approx 2.6$ square inches.
3) By iteration, $P_3 = (¾)^3 A_0 = (¾)^3 \times 6 \approx 1.46$ square inches
f) The area of the completed figure is **zero**.

4. $(3-4i)-(2+5i)=1-9i$

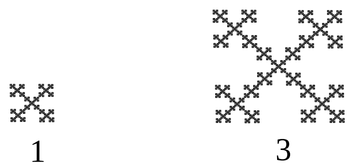
5. Multiply $(3-4i)(2+3i)=6+9i-8i-12i^2 = 6+i-12(-1)=18+i$

6. Generate terms:

$$z_1 = z_0^{\,2} + 0.4 + 0.3i = 0 + 0.4 + 0.3i = 0.4 + 0.3i$$
$$z_2 = z_1^{\,2} + 0.4 + 0.3i = (0.4+0.3i)^2 + 0.4 + 0.3i =$$
$$z_3 = z_2^{\,2} + 0.5i = (-0.25+0.5i)^2 + 0.5i = -0.1875 + 0.25i$$
$$z_4 = z_3^{\,2} + 0.5i = (-0.1875+0.25i)^2 + 0.5i = -0.02734 + 0.40625\,i$$

From these values, it does appear that this terms of the sequence stay within a radius of 2 from the origin, suggesting that $c = 0.4 + 0.3i$ is part of the Mandelbrot set.

7. Scale the fractal up to three times its original size:



Scaling the fractal by a factor of 3 requires 5 copies of the original, so $D = \dfrac{\log(5)}{\log(3)} \approx 1.465$

Additional Resources

A much more extensive coverage of fractals can be found on the Fractal Geometry site. This site includes links to several Java software programs for exploring fractals.

The Mandelbrot Explorer site, provides more details on the Mandelbrot set, including a nice visualization of Mandelbrot sequences.

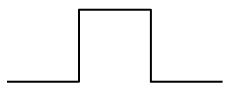
Chapter 6 Exercises

Iterated Fractals

Using the initiator and generator shown, draw the next two stages of the iterated fractal.

1.

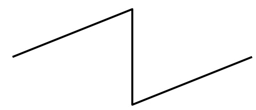
initiator



generator

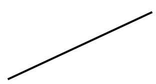
2.

initiator

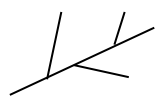


generator

3.



initiator



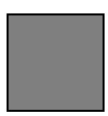
generator

4.

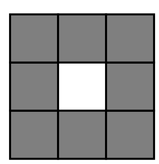
initiator

generator

5.

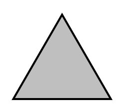


initiator

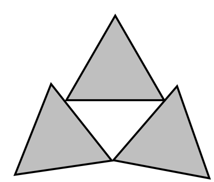


generator

6.



initiator



generator

7. Create your own version of Sierpinski gasket with added randomness.

8. Create a version of the branching tree fractal from example #3 with added randomness.

Koch Snowflake and Sierpinski Gasket

9. Suppose that a Koch snowflake is being constructed, and that the initial equilateral triangle has perimeter or 9 inches and area of 7.79 square inches.

   a) What is the perimeter of the figure after the first step of the construction?
   b) What is the perimeter of the figure after the fourth step of the construction?
   c) What is the area of the completed Koch snowflake?

10. Suppose that a Koch snowflake is being constructed, and that the figure at the second iteration has a perimeter of 16 inches.

    a) What was the perimeter of the starting triangle?
    b) What is the perimeter of the figure after the fifth step of the construction?
    c) What is the perimeter of the completed Koch snowflake?

11. Suppose that a Sierpinski gasket is being constructed, and that the initial equilateral triangle has area of 16 square inches.

    a) What is the area of the figure after the first step of the construction?
    b) What is the area of the figure after the fourth step of the construction?
    c) What is the area of the completed Sierpinski gasket?

12. Suppose that a Sierpinski gasket is being constructed, and that the figure at the second iteration has a perimeter of 18 inches.

    a) What was the perimeter of the starting triangle?
    b) What is the perimeter of the figure after the fifth step of the construction?
    c) What is the perimeter of the completed Sierpinski gasket?

Complex Numbers

13. Plot each number in the complex plane: a) 4 b) $-3i$ c) $-2+3i$ d) $2+i$

14. Plot each number in the complex plane: a) -2 b) $4i$ c) $1+2i$ d) $-1-i$

15. Compute: a) $(2+3i)+(3-4i)$ b) $(3-5i)-(-2-i)$

16. Compute: a) $(1-i)+(2+4i)$ b) $(-2-3i)-(4-2i)$

17. Multiply: a) $3(2+4i)$ b) $(2i)(-1-5i)$ c) $(2-4i)(1+3i)$

18. Multiply: a) $2(-1+3i)$ b) $(3i)(2-6i)$ c) $(1-i)(2+5i)$

19. Plot the number $2+3i$. Does multiplying by $1-i$ move the point closer to or further from the origin? Does it rotate the point, and if so which direction?

20. Plot the number $2+3i$. Does multiplying by $0.75+0.5i$ move the point closer to or further from the origin? Does it rotate the point, and if so which direction?

Recursive Sequences

21. Given the recursive relationship $z_{n+1} = iz_n + 1, \quad z_0 = 2$, generate the next 3 terms of the recursive sequence.

22. Given the recursive relationship $z_{n+1} = 2z_n + i, \quad z_0 = 3 - 2i$, generate the next 3 terms of the recursive sequence.

23. Using $c = -0.25$, calculate the first 4 terms of the Mandelbrot sequence.

24. Using $c = 1 - i$, calculate the first 4 terms of the Mandelbrot sequence.

For a given value of c, the Mandelbrot sequence can be described as *escaping* (growing large), a *attracted* (it approaches a fixed value), or *periodic* (it jumps between several fixed values). A periodic cycle is typically described the number if values it jumps between; a 2-cycle jumps between 2 values, and a 4-cycle jumps between 4 values.

For questions 25 – 30, you'll want to use a calculator that can compute with complex numbers, or use an online calculator which can compute a Mandelbrot sequence. For each value of c, examine the Mandelbrot sequence and determine if the value appears to be escaping, attracted, or periodic?

25. $c = -0.5 + 0.25i$.

26. $c = 0.25 + 0.25i$.

27. $c = -1.2$.

28. $c = i$.

29. $c = 0.5 + 0.25i$.

30. $c = -0.5 + 0.5i$.

31. $c = -0.12 + 0.75i$.

32. $c = -0.5 + 0.5i$.

Fractal Dimension

33. Determine the fractal dimension of the Koch curve.

34. Determine the fractal dimension of the curve generated in exercise #1

35. Determine the fractal dimension of the Sierpinski carpet generated in exercise #5

36. Determine the fractal dimension of the Cantor set generated in exercise #4

Exploration

The Julia Set for *c* is another fractal, related to the Mandelbrot set. The Julia Set for *c* uses the recursive sequence: $z_{n+1} = z_n^{\ 2} + c, \quad z_0 = d$, where *c* is constant for any particular Julia set, and *d* is the number being tested. A value *d* is part of the Julia Set for *c* if the sequence does not grow large.

For example, the Julia Set for -2 would be defined by $z_{n+1} = z_n^{\ 2} - 2, \quad z_0 = d$. We then pick values for *d*, and test each to determine if it is part of the Julia Set for -2. If so, we color black the point in the complex plane corresponding with the number *d*. If not, we can color the point *d* based on how fast it grows, like we did with the Mandelbrot Set.

For questions 33-34, you will probably want to use the online calculator again.

37. Determine which of these numbers are in the Julia Set at $c = -0.12i + 0.75i$
    a) $0.25i$ b) 0.1 c) $0.25 + 0.25i$

38. Determine which of these numbers are in the Julia Set at $c = -0.75$
    a) $0.5i$ b) 1 c) $0.5 - 0.25i$

You can find many images online of various Julia Sets[11].

39. Explain why no point with initial distance from the origin greater than 2 will be part of the Mandelbrot set.

[11] For example, http://www.jcu.edu/math/faculty/spitz/juliaset/juliaset.htm,

CHAPTER 7

Symmetry and the Golden Ratio

Excerpted from *College Mathematics for Everyday Life*
by Inigo, Jameson, Kozak, Lanzetta and Sonier

Symmetry and the Golden Ratio

Patterns and geometry occur in nature and people have been noticing these patterns since the dawn of humanity. In this chapter, topics in geometry will be examined. These topics include transformation and symmetry of geometric shapes, similar figures, gnomons, Fibonacci numbers, and the Golden Ratio.

Section 7.1: Transformations Using Rigid Motions

In this section we will learn about isometry or *rigid motions*. An isometry is a transformation that preserves the distances between the vertices of a shape. A rigid motion does not affect the overall shape of an object but moves an object from a starting location to an ending location. The resultant figure is congruent to the original figure.

A **rigid motion** is a transformation that moves an object from one location to another, while keeping the size and shape of the object unchanged.

Two figures are **congruent** if and only if there exists a rigid motion that sets up a correspondence of one figure as the image of the other. Side lengths remain the same and interior angles remain the same.

The **identity motion** is a rigid motion that moves an object from its starting location to exactly the same location. It is as if the object has not moved at all.

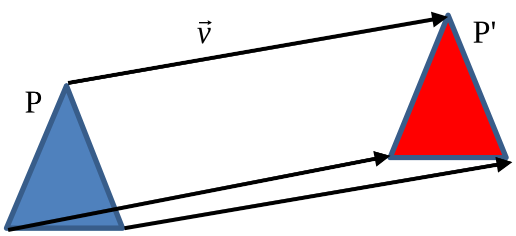
Given a rigid motion T, the **inverse motion** is a rigid motion T^{-1} such that T followed by T^{-1} results in the identity motion. That is, the inverse motion moves the object back to its starting location.

There are four kinds of rigid motions: translations, rotations, reflections, and glide-reflections. When describing a rigid motion, we will use points like P and Q, located on the geometric shape, and identify their new location on the moved geometric shape by P' and Q'.

We will start with the rigid motion called a translation. When translating an object, we move the object in a specific direction for a specific length. The direction and length of the motion is usually described by a directed line segment $\vec{v}$, called a vector.

Definition: A **translation** of an object moves the object along a directed line segment called a vector $\vec{v}$ for a specific distance and in a specific direction. The motion is completely determined by two points P and P' where P is on the original object and P' is on the translated object.

Figure 1: Translation

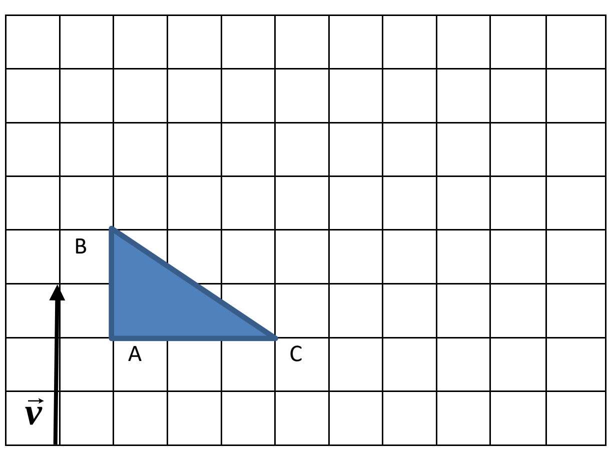


The translation of the blue triangle with point P was moved along the vector $\vec{v}$ to the location of the red triangle with point P'. Also note that the other vertices of the blue triangle also moved along the vector $\vec{v}$ to corresponding vertices on the red triangle.

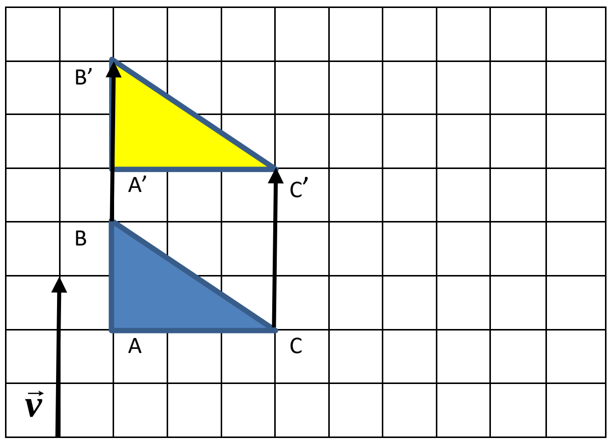
In ordinary language, a translation of an object is a *slide* from one position to another. You are given a geometric figure and an arrow which represents the vector. The vector gives you the direction and distance which you slide the figure.

Example 7.1.1 Translation of a Triangle

Given the blue triangle and vector $\vec{v}$ shown below, find the image of the triangle for a translation along vector $\vec{v}$.



The vector $\vec{v}$ extends vertically for 3 units. So, we start by moving each vertex of the triangle vertically for 3 units. Then we draw the edges connecting the vertices, to get the image, shown as a yellow triangle.



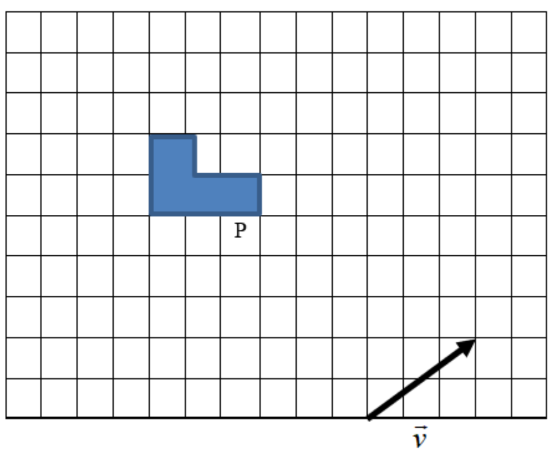
Note that when we moved the triangle, *all* of the points on the triangle moved – not just the vertices, but also all the points along the edges. That is, there are no fixed points. Moreover, if we wanted to move the triangle back to its original location, we would just have to translate it 3 units in the opposite direction – that is 3 units straight down. More generally, we have:

Properties of a Translation

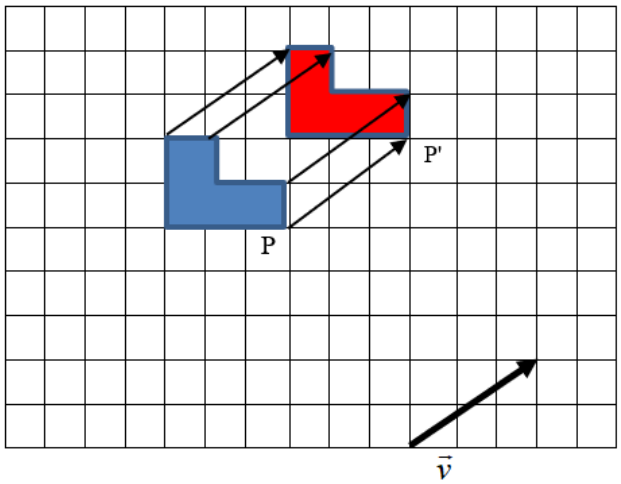
1. A translation is completely determined by two points P and P′.
2. A translation has no fixed points.
3. The inverse motion is a translation along the vector $-\vec{v}$, where $-\vec{v}$ has the same length as vector $\vec{v}$, but points in the opposite direction.

Example 7.1.2 Translation of an "L-Shaped" Object

Given the L-shape figure below, translate the figure along the given vector $\vec{v}$.



The vector $\vec{v}$ moves horizontally three units to the right and vertically two units up. So, we start by moving each vertex three units to the right and two units up, and then fill in the edges. The red figure is the position of the L-shape figure after the translation.



The next type of transformation (rigid motion) that we will discuss is called a *rotation*.

> Definition: A **rotation** of an object moves the object around a point called the rotocenter R a specified angle θ either clockwise or counterclockwise.

NOTE: The rotocenter R can be outside the object, inside the object or on the object. In the two figures below, the blue triangle below has been by a specified angle about a point R. In Figure 2, the rotocenter is outside the object, but in Figure 3, the rotocenter is inside the triangle.

Figure 2: Triangle Rotated 90° around Rotocenter R outside the Triangle

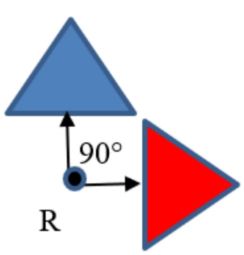
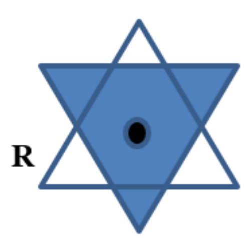


Figure 3: Triangle Rotated 180° around Rotocenter R inside the Triangle



Note that when we perform a rotation, the rotocenter remains fixed; and it is the only fixed point for this transformation. Moreover, if we rotate by 360°, the figure will come back to its starting point. This means that a rotation by 360° is the *identity motion*.

Another key property of rotations is that they are *additive*. That is, if we do a rotation by α° followed by a rotation by β°, then this is the same as doing a single rotation by the sum of the two angles, $(\alpha + \beta)$°. In particular, if we rotate by α° followed by a rotation by $(360 - \alpha)$°, the result is a rotation by 360° which is the identity. Thus, the inverse of a rotation by α° is a rotation by $(360 - \alpha)$°.

Finally, every rotation can be described by an angle between 0° and 360°. For example, if we did a rotation of 405°, this would be the same as rotating by 360° followed by a rotation if 45°. But since rotation by 360° is the identity motion, this would be the same as if we just rotated by 45°.

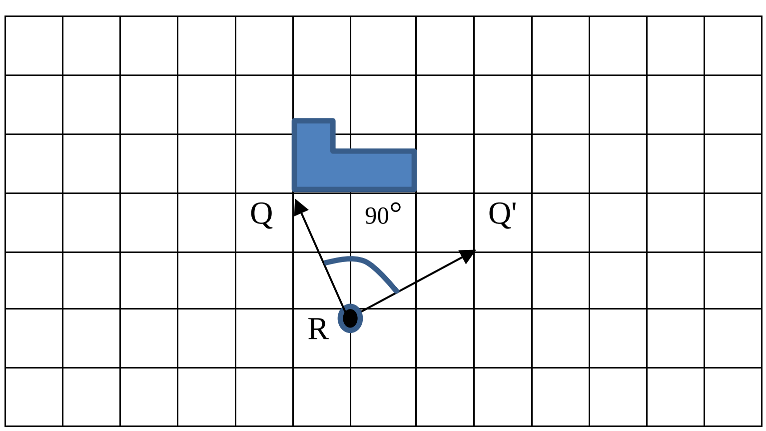
We summarize the key properties below:

Properties of a Rotation

1. A rotation is completely determined by two pairs of points; P and P′ and Q and Q′
2. A rotation has one fixed point, the rotocenter R
3. The identity motion is a 360° rotation
4. The inverse of a clockwise rotation by α° is a clockwise rotation by $(360 - \alpha)$°. Alternately, the inverse of a clockwise rotation by α° is a *counter-clockwise* rotation by α°.

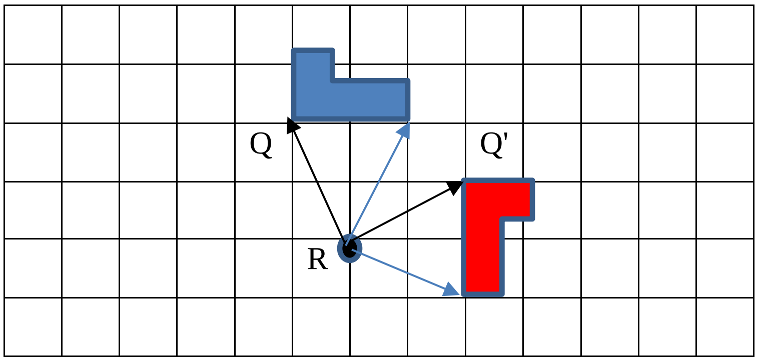
Example 7.1.3: Rotation of an L-Shaped Object

Rotate the L-shaped figure in the diagram below 90° clockwise about the rotocenter R.



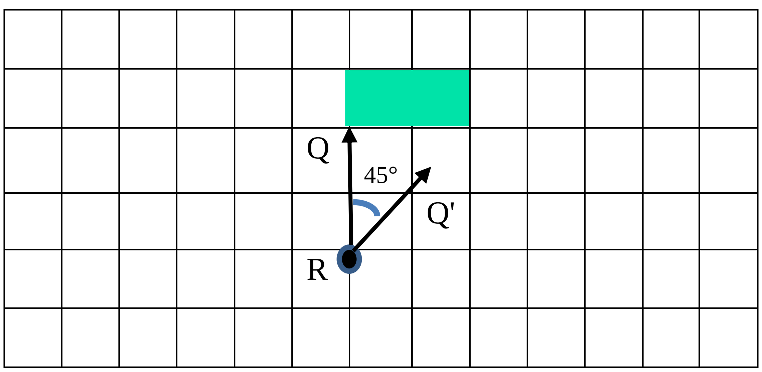
The point Q (bottom left vertex of the figure) is rotated 90°.
Note that Q is one unit left and two units up from R. After the rotation, the image Q′ is one unit up and two units right from R.

Now, rotate each of the other vertices 90° clockwise as well, and then fill in the edges. The red L-shaped figure will be the result; i.e. the red L-shape is the image of the original blue figure when rotated 90° clockwise.

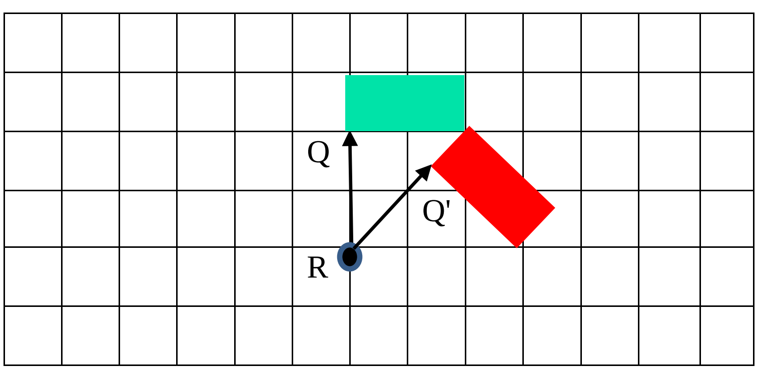


Example 7.1.4: 45° Clockwise Rotation of a Rectangle

Rotate the green triangle 45° clockwise about the rotocenter R shown.

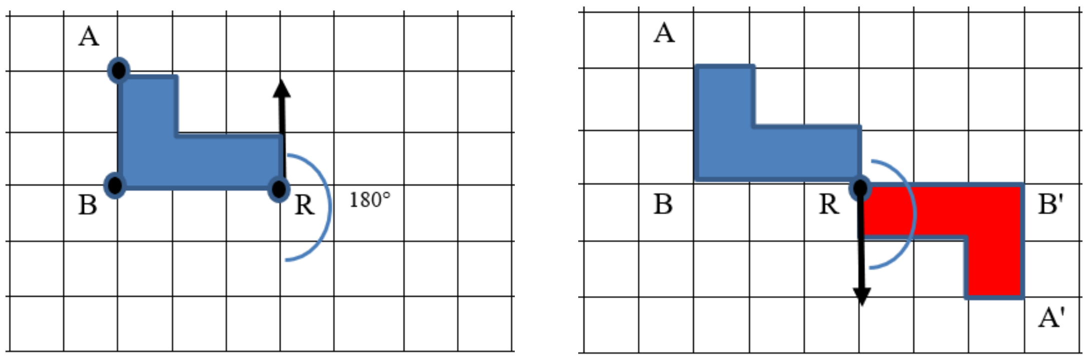


Again, rotate all four vertices by 45° and fill in the edges to get the red rectangle shown.



Example 7.1.5: 180° Clockwise Rotation of an L-Shape

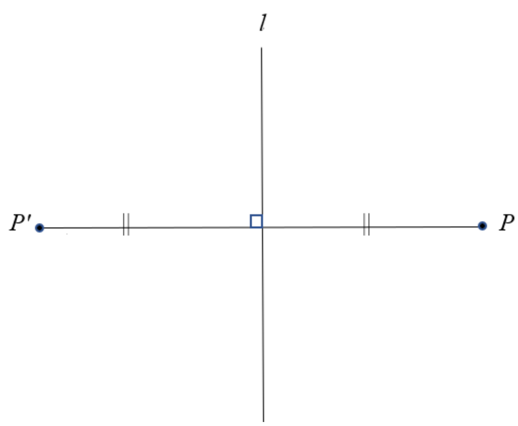
Rotating the blue L-Shape on the left by 180° about the rotocenter R moves the object to the position of the red figure on the right. Note that the rotocenter (the bottom right vertex of the figure) remains fixed.



The next type of transformation (rigid motion) is called a reflection. A reflection is a mirror image of an object, or can be thought of as "flipping" an object over.

> Definition**:** Let l be a line in the plane. Then the **reflection across *l*** is a rigid motion that sends any point P to the unique point P' such l is the perpendicular bisector of PP'.
>
> The line l is called either the *line of reflection* or the *axis* of the reflection.

Figure 4: Reflection across a line *l*.



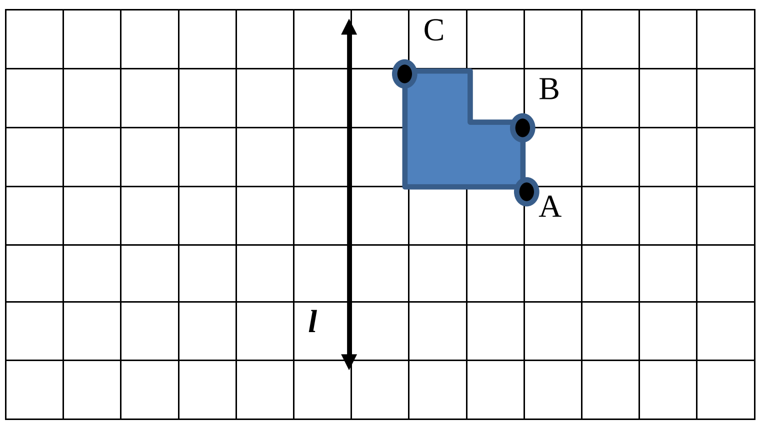
In plain English, we might describe a reflection as a mirror image across a line l. From the definition, we see that any point on the axis l is a fixed point for this rigid motion. Moreover, if we know the line l, then we can determine the image of any point; conversely if we know the image P' of a single point P that is not on the line, we can determine the line (by constructing the perpendicular bisector of PP'). Finally, if we do the reflection twice in succession, we see that every point is back where it started. That is, reflection across a line l is its own inverse.

Properties of a Reflection

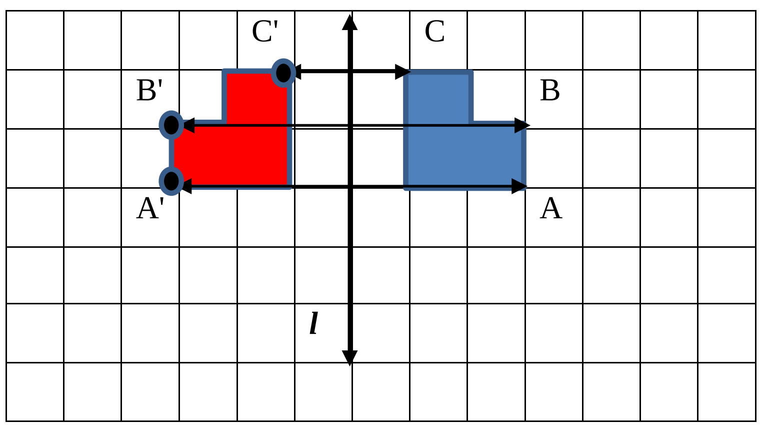
1. A reflection is completely determined by a single pair of points; P and P’
2. A reflection is completely determined by its axis *l*.
3. A reflection has infinitely many fixed points: every point on the line of reflection *l* remains fixed by the motion.
4. A reflection is its own inverse. That is, performing the reflection twice in succession will return the object to its starting location.

Example 7.1.6: Reflection of an Object about a Line *l*

Reflect the object on the right across the line *l*.

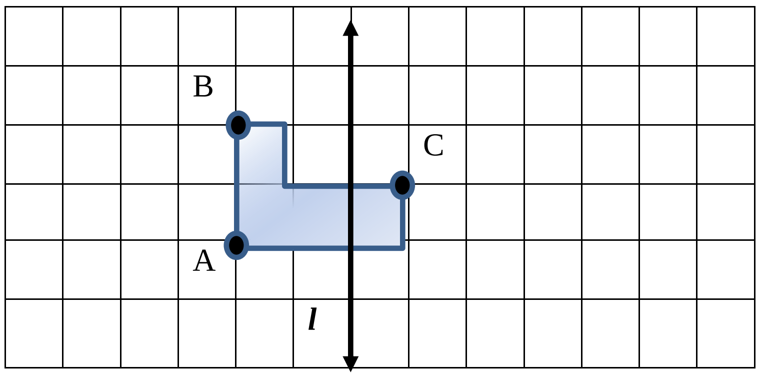


The reflection places each vertex along a line perpendicular to *l* and equidistant from *l*. After plotting the images A′, B′ and C′, fill in the edges to get the image of the object (in red) on the left:

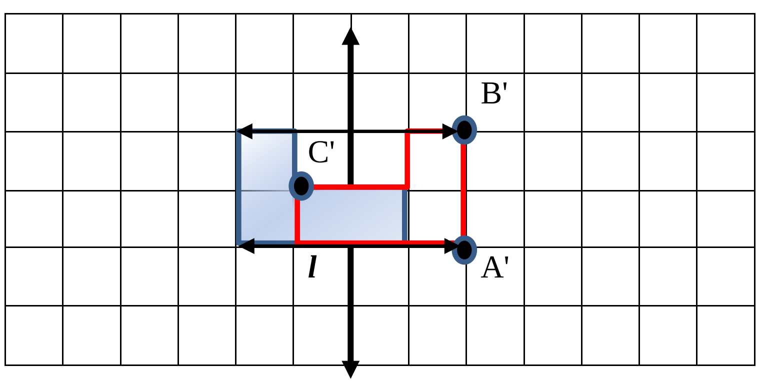


Example 7.1.7: Reflect an L-Shape across a Line *l*

Reflect the L-Shape across the line *l* shown in the diagram:



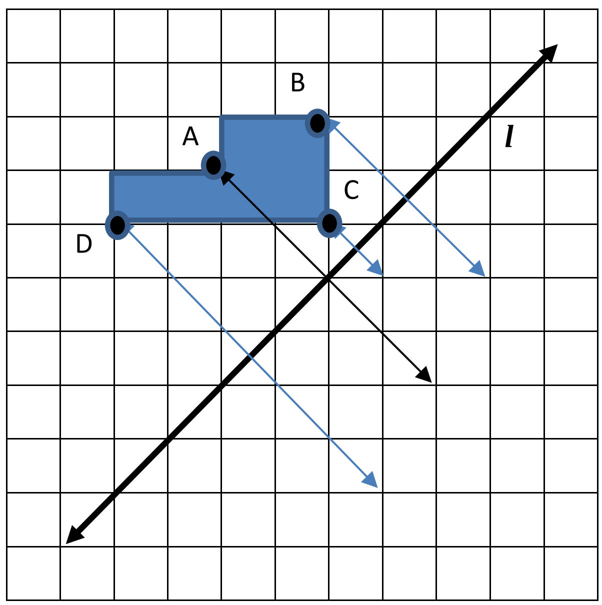
This one is slightly different, as the line of reflection passes through the figure. However, the strategy is the same – we reflect vertices A, B and C, and fill in the edges between the images A′, B′ and C′. The red L-shape shown below is the result after the reflection. Since the line of reflection is vertical, the original position of each vertex is on a horizontal line with the reflected position of each vertex. (This line that connects the original and reflected positions of the vertex is perpendicular to line *l* and the original and reflected positions of each vertex are equidistant to line *l*.)



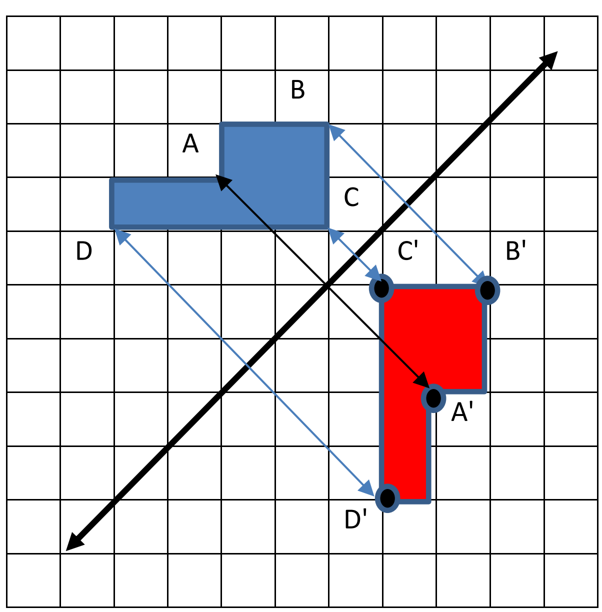
Example 7.1.8: Reflect another L-Shape across Line *l*

Reflect the L-Shape across the line *l* shown in the diagram.

First we identify and label the vertices of the figure. From each vertex, draw a line segment perpendicular to line *l* and make sure its midpoint lies on line *l*.



Now draw the new positions of the vertices, and fill in the edges. The result (in red) is mirror image of the original figure.

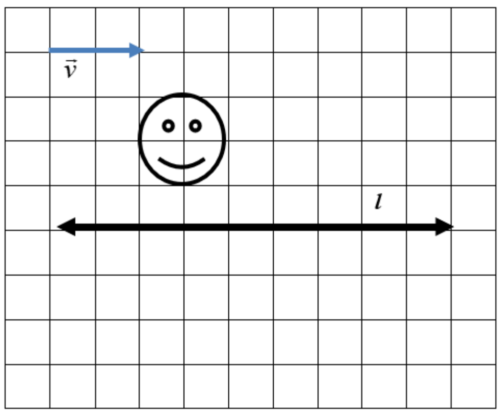


The final transformation (rigid motion) that we will study is a glide-reflection, which is simply a combination of two of the other rigid motions.

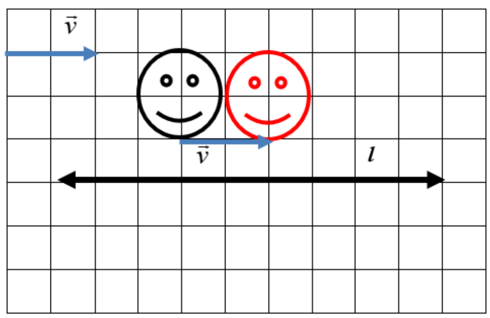
A **glide-reflection** is a combination of a reflection and a translation.

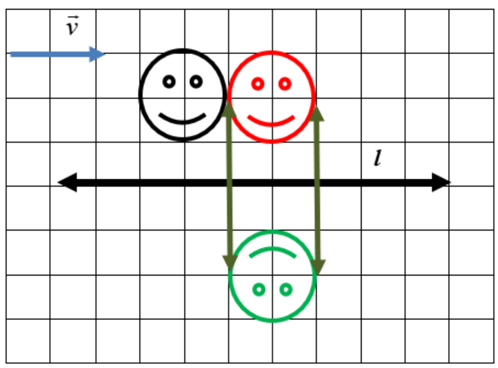
Example 7.1.9: Glide-Reflection of a Smiley Face by Vector $\vec{v}$ and Line *l*

Find the image of the figure shown under the glide reflection using vector $\vec{v}$ and line l.



This is a two-step process. First slide the smiley face two units to the right along the vector $\vec{v}$; then reflect the smiley face across line l. The final result is the green upside-down smiley face. The final result is the green upside-down smiley face



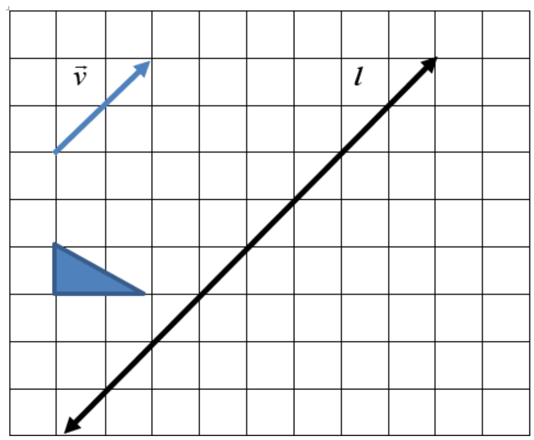


Properties of a Glide-Reflection

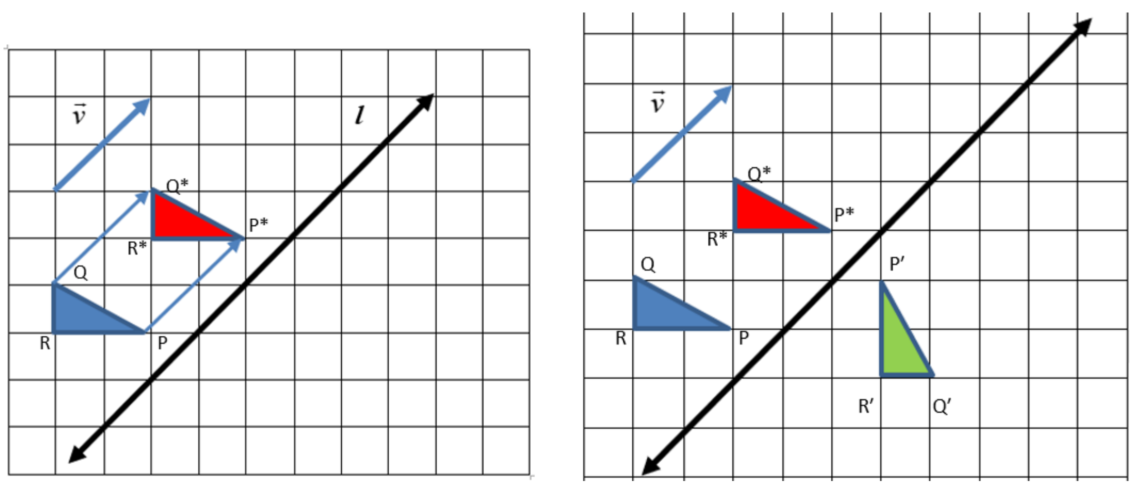
1. A glide reflection is completely determined by a single pair of points; P and P′.
2. Every point on the line of reflection l is a fixed point.
3. The inverse motion is the reverse glide-reflection; that is, the combination of translation by $-\vec{v}$ followed by reflection across the line l.

Example 7.1.9: Glide-Reflection of a Triangle

Find the image of the triangle under the glide reflection using vector $\vec{v}$ and line l:

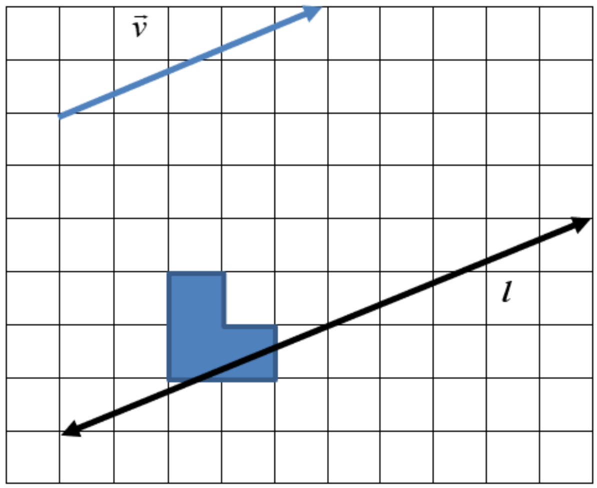


Again, this is a two-step procedure. First, slide the triangle along vector $\vec{v}$ (result is shown in red). Then, reflect the triangle across line l. The final result is the green triangle below line l.

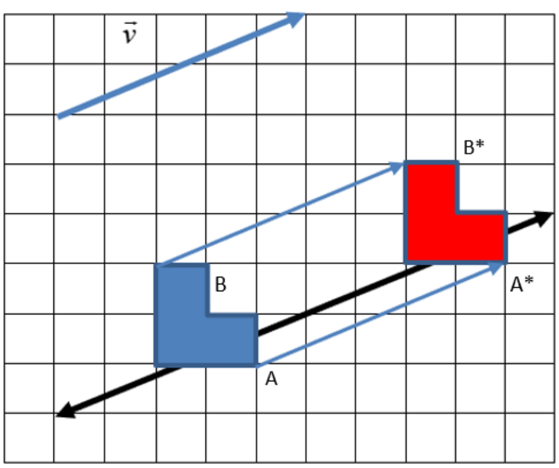


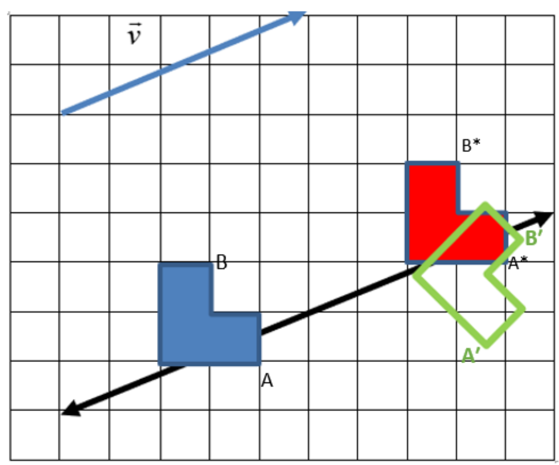
Example 7.1.10: Glide-Reflection of an L-Shape

Find the image of the figure under the glide reflection using vector $\vec{v}$ and line l:



Like the previous examples, there are two steps. First slide the L-shape along vector $\vec{v}$, which gives the red shape on the left. Then reflect the L-shape across line l, to get the open green shape on the right.





Section 7.2: Rigid Motions and Symmetry

Humans have long associated symmetry with beauty and art. In this section, we give a mathematical definition of symmetry in terms of rigid motions.

Definition: A **symmetry** of an object is a rigid motion that moves an object back onto itself.

There are two categories of symmetry in two dimensions, reflection symmetries and rotation symmetries.

A **reflection symmetry** occurs when an object has a line of symmetry going through the center of the object, and you can fold the object on this line and the two halves will "match." An object may have no reflection symmetry or may have one or more reflection symmetries.

A **rotation symmetry** occurs when an object has a rotocenter in the center of the object, and the object can be rotated about the rotocenter some degree less than or equal to 360° and is a "match" to the original object. Every object has one or more rotation symmetries.

D-Type Symmetry: Objects that have both reflection symmetries and rotation symmetries are Type D_n where n is either the number of reflection symmetries or the number of rotation symmetries. (The D is for "*dihedral symmetry*", which is the formal mathematical term to describe this particular kind of symmetry.)

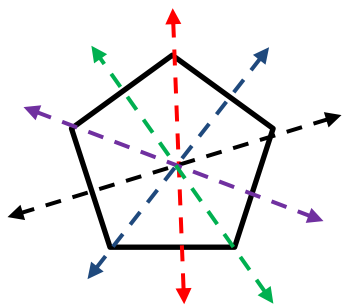
If an object has both reflection and rotation symmetries, then it is always the same number n of each kind of symmetry.

Z-Type Symmetry: Objects that have no reflection symmetries and only rotation symmetries are Type Z_n where n is the number of the rotation symmetries.

Example 7.2.1: Symmetries of a Pentagon

We will identify the reflection and rotation symmetries of a regular pentagon (meaning that all side lengths are equal and all angles are equal).

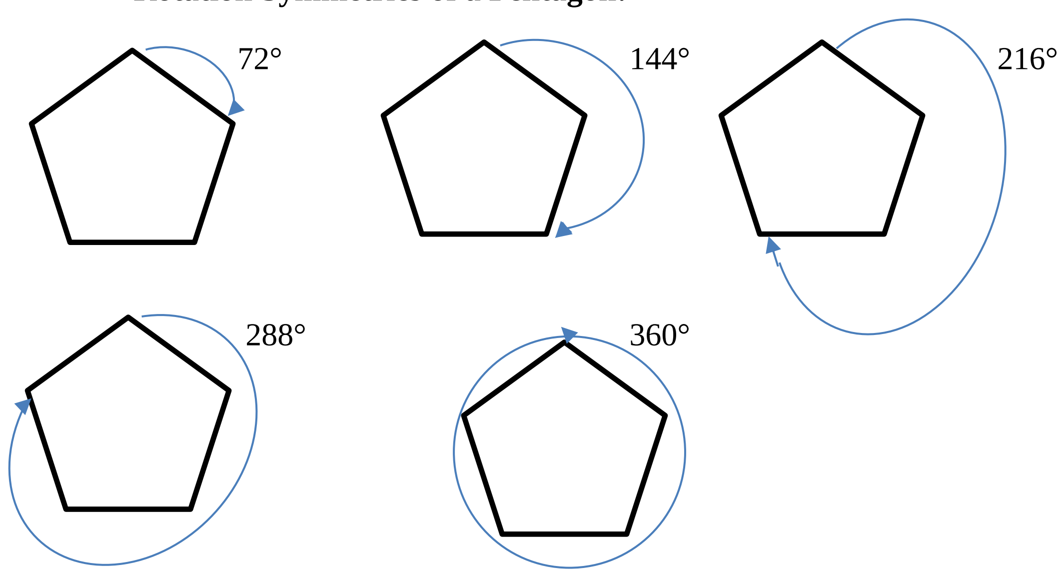
The five dashed lines shown on the figure below are lines of reflection. Each line passes through one of the vertices, and bisects the side opposite to that vertex. The pentagon can then be folded along any of these lines back onto itself and the two halves will "match" which means that the pentagon has a reflection symmetry along each line.



There are also five rotation symmetries.

If we divide 360° by the number of vertices, we get $\frac{360^{\circ}}{5} = 72^{\circ}$. So, if we rotate pentagon clockwise by 72°, each vertex will land in the position of another vertex. That is, the resulting object will match the original object. Thus, clockwise rotation by 72° is a rotational symmetry. Similarly, rotating by a multiple of 72°, we get another rotation symmetry; of course, if we rotate by 72° five times in succession, we will have rotated by a full 360° and will be back at the starting point.

Rotation Symmetries of a Pentagon:



Recall that when an object has the same number of reflection symmetries as rotation symmetries, we say it has symmetry type D_n. Therefore, the pentagon has symmetry type D_5 because it has five reflection symmetries and five rotation symmetries.

A regular polygon is a polygon that has n equal sides and n equal angles.
For example, when $n = 3$, we get an equilateral triangle. When $n = 4$, we get a square, for $n = 6$, we get a regular hexagon, etc. All of these figures have type D symmetries.

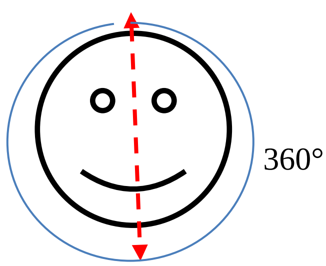
More precisely, we have the following general fact:

Theorem: A regular polygon with n vertices has symmetry type D_n.

Example 7.2.2: Symmetries of a Smiley Face

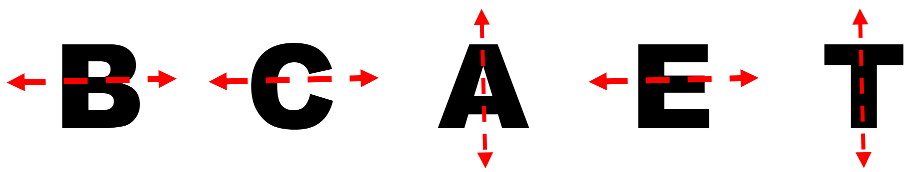
Next we identify the rotation and the reflection symmetries of the smiley face.

There is one line of reflection that will produce a reflection symmetry as shown below. And if there is only one reflection there can be only one rotation symmetry as well; of course that would be rotation by 360° (which every figure has as a symmetry).



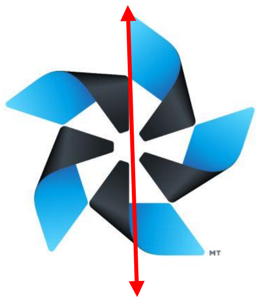
Example 7.2.3: Letters with Symmetry Type D_1

The following letters are all examples of symmetry type D_1 since they each have only one axis of reflection that will produce a symmetry as shown below, and they each have only one rotation symmetry, 360°.



Example 7.2.4: Symmetries of a Pinwheel

Identify the rotation and reflection symmetries of a pinwheel.



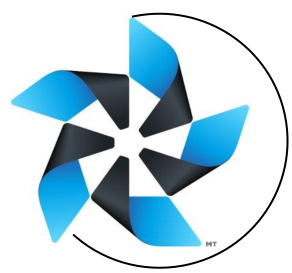
There are no reflection symmetries of the pinwheel. Because of the shape of the blades of the pinwheel, there is no way to draw a line through the figure that would give us a reflection symmetry.

However, there are five rotation symmetries.
There are five identical blades, and one fifth of 360º is 72º. So if we rotate clockwise by 72º, each blade will land exactly in the position of one a previous blade; thus rotation by 72º is a symmetry. And just like with the pentagon, repeatedly rotating by 72º is again a symmetry.

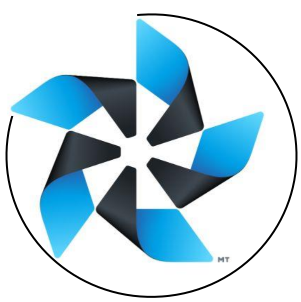
Thus, the rotation symmetries of the pinwheel are 72°, 144°, 216°, 288°, and 360°.

72°

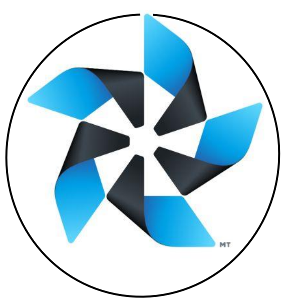
144°



216°



288°



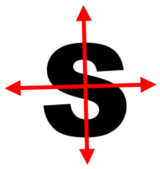
360°

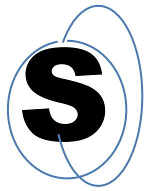
When an object has no reflection symmetries and only rotation symmetries, we say it has symmetry type Z_n. The pinwheel has symmetry type Z_5.

Example 7.2.5: Symmetries of the Letter S

Identify the rotation and the reflection symmetries of the letter S, and determine its symmetry type.

There are no reflection symmetries and two rotation symmetries; 180° and 360°, therefore the letter S has symmetry type Z_2.



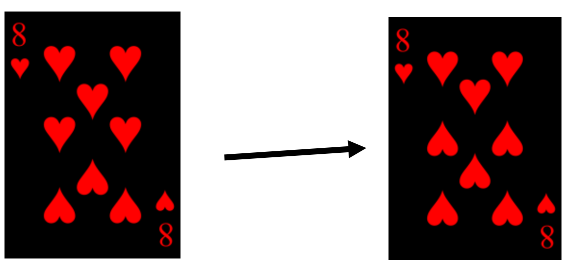


Example 7.2.6: Symmetries of the Card the Eight of Hearts

Identify the rotation and reflection symmetries of card the eight of hearts.

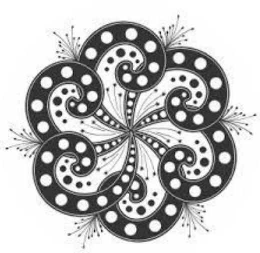
The card shown below has no reflection symmetries since any reflection would change the orientation of the card.

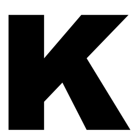
At first glance, it may appear that the card has symmetry type Z_2. However, when rotated 180°, the top five hearts will turn upside-down and it will not be the same. Therefore, this card has only the 360° rotation symmetry, and so it has symmetry type Z_1.



Example 7.2.7: Other Figures of Symmetry Type Z_n.

Identify the symmetry type of each figure:

a. 

b. 

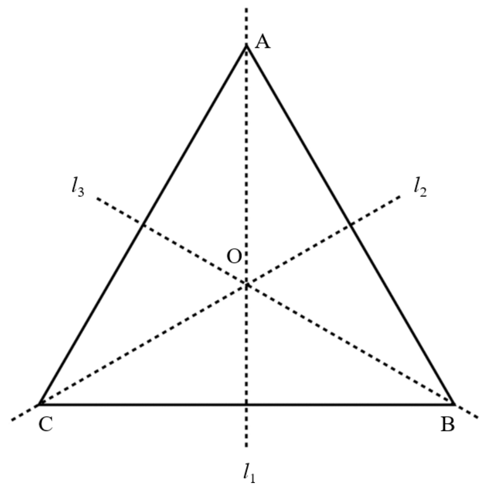
a. The design has symmetry type Z_6, no reflection symmetries and six rotation symmetries. To find the degrees for the rotation symmetries, divide 360° by the number of "points" in the figure: $\frac{360°}{6} = 60°$. The six rotation symmetries are multiples of 60°:

60°, 120°, 180°, 240°, 300°, and 360°.

b. The letter K has symmetry type Z_1, no reflection symmetries and one rotation symmetry (360°).

Example 7.2.8: **Symmetries of an Equilateral Triangle**.

We have already noted that an equilateral triangle will have symmetry type D_3, meaning that there are 3 rotation symmetries and 3 reflection symmetries. In this example, we will investigate these symmetries in more detail. Suppose we have the triangle:



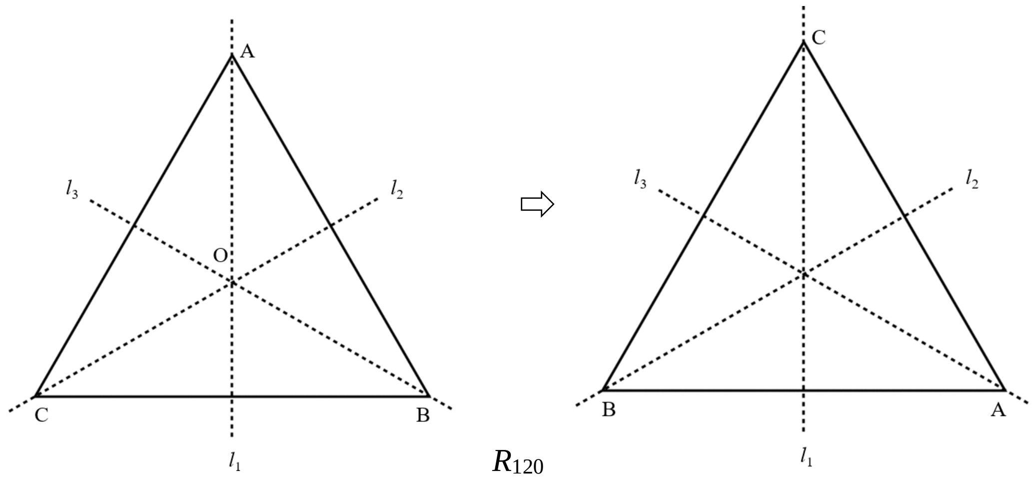
Then there are three rotational symmetries, with rotocenter O. We label them as follows:

R_{120} ⇔ Clockwise rotation by 120°
R_{240} ⇔ Clockwise rotation by 240°
R_{360} ⇔ Clockwise rotation by 360° (the identity transformation)

And there are three reflection symmetries, which we label as follows:

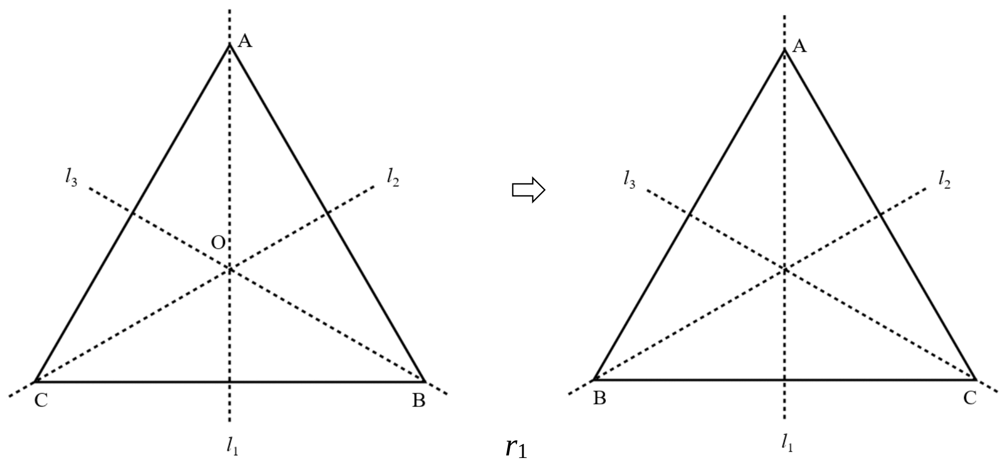
r_1 ⇔ Reflection across l_1
r_2 ⇔ Reflection across l_2
r_3 ⇔ Reflection across l_3

By definition, when we apply one of the rigid motions above, the figure will land back on itself. However, the vertices themselves will not be in their original positions. For example, if we rotate by 120^o, each vertex moves one notch clockwise (note that the dotted lines in the background do *not* rotate):



R_{120}

Similarly, if we reflect the figure about the line l_1, the point A will remain fixed, and vertices B and C will change places.

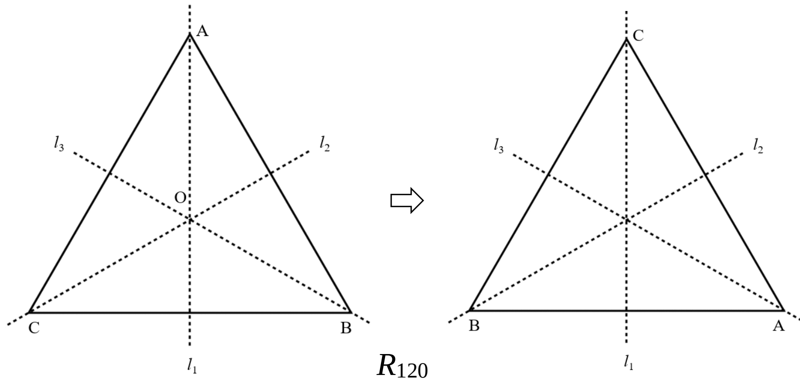


r_1

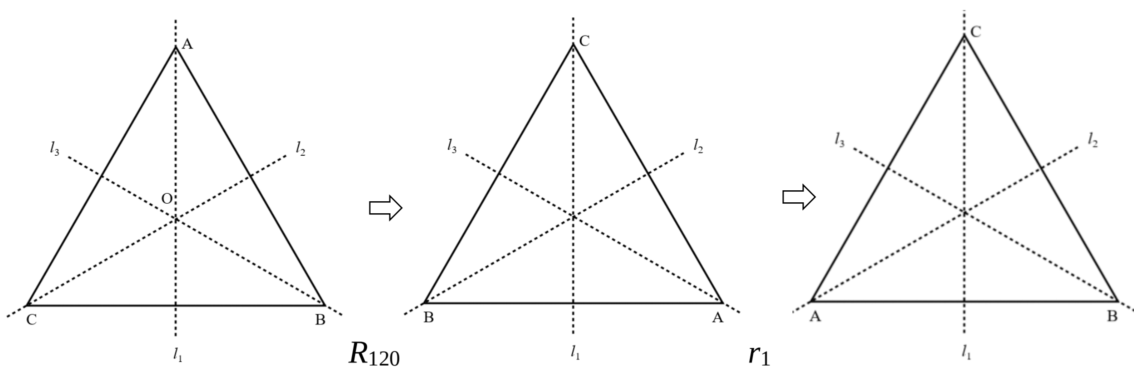
Now we examine what happens when we follow one of these rigid motions by another. We know that the result must be another symmetry, since at each stage the figure lands back on itself.

We already know what happens when we follow a rotation by another rotation: we simply add the angles of rotation. For example, when we rotate by 120^o twice in succession, each vertex will move two notches clockwise, which is equivalent to rotating the figure by 240^o. In other words, when we follow R_{120} by R_{120}, we get R_{240}.But what happens when we follow a rotation by a reflection? Or vice versa?

Let’s try following rotation by 120° with reflection across l_1. First, the rotation:

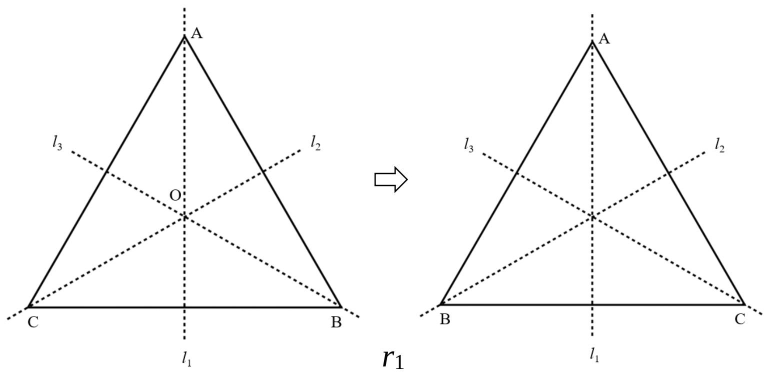


Next, the reflection: Since vertex C is now on the line l_1, it will stay fixed for this motion. Vertices A and B will switch places:

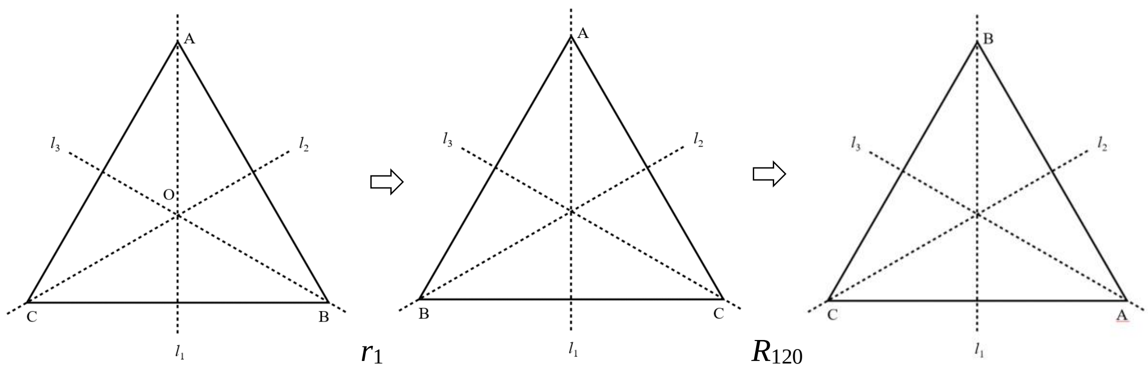


The result of this process must be another symmetry, but which one? Observe that vertex B is back in its original position; and since this symmetry has a fixed point, it cannot be a rotation. And we deduce that it must be reflection across line l_3 since that is the line passing through B in the original picture a moreover, reflection across l_3 would switch the positions of A and C, which is exactly what happened.

Next, we'll do the same two operations, but in the opposite order. That is, we'll do reflection across l_1 followed by rotation of 120^o. First the reflection:



Now, we do the rotation. In this step, each vertex will move around one notch clockwise:



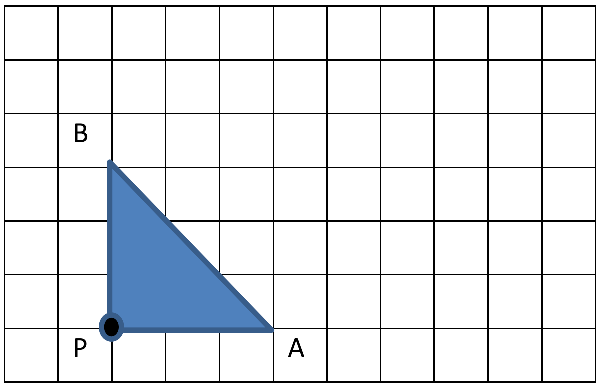
Again, the result of this process must be another symmetry. To identify which one, note that the overall result was to keep C fixed, while switching the positions of A and B. In other words, reflection across l_1 followed by rotation by 120^o is the same as reflection across l_2.

Observe that these two calculations gave us different answers – the order in which we carry out the rigid motions matters.

Section 7.3: Scaling Transformations and Similar Figures

This section covers transformations that either enlarge or shrink an object from a point, P. Such transformations are called **size transformations** or **scaling transformations.** The point P is called the **center** of the size transformation. The multiplier used to enlarge or shrink the object is called the **scale factor**, *k*. The calculations used to make the transformation depend on the distances from point P to the vertices of the object. These distances are multiplied by *k*. So, to enlarge or shrink an object, find the distance from the point P to a vertex A of the object. Multiply this distance by k to get $k \cdot \text{PA}$, where PA represents the distance from P to A. Then, measure this new distance, $k \cdot \text{PA}$, from point P in the direction of vertex A. This distance gives the new location of vertex A after the object has been size-transformed. Repeat for all vertices of the object.

Example 7.3.1: Enlarge a Triangle by a Factor of Two



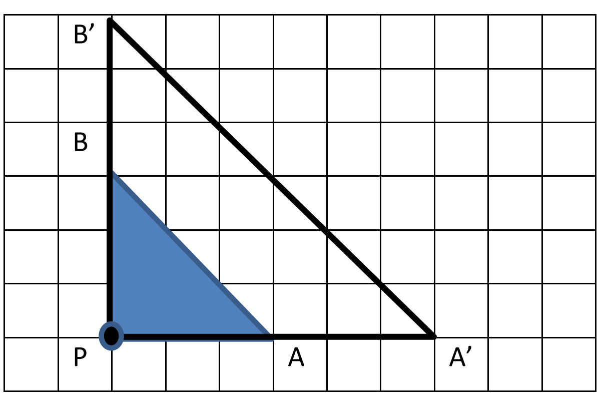
Step 1: Measure the distances from point P to each vertex of the triangle. One vertex of the triangle is on point P, so that vertex will remain at point P.

The distance from point P to vertex A is three units.
The distance from point P to vertex B is also three units.

Step 2: Multiply these distance by the scale factor two.

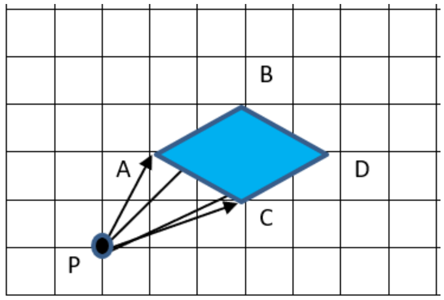
$$2 \cdot \text{PA} = 2(3) = 6$$
$$2 \cdot \text{PB} = 2(3) = 6$$

Step 3: Measure six units from point P in the direction of vertex A and measure six units from point P in the direction of vertex B. The new locations of A and B are each six units from P in their corresponding directions as shown below.

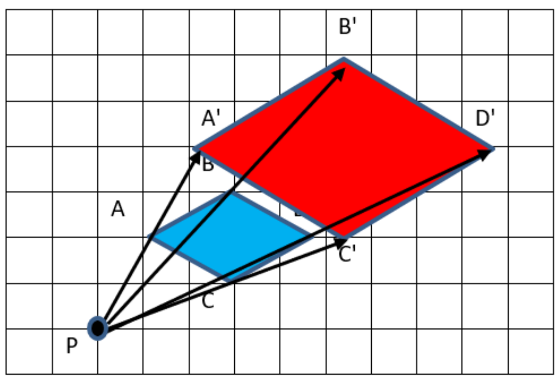


Example 7.3.2: Enlarge a Diamond by a Factor of Two

Scale the four sided "diamond" figure by a factor of 2, using P as the center.

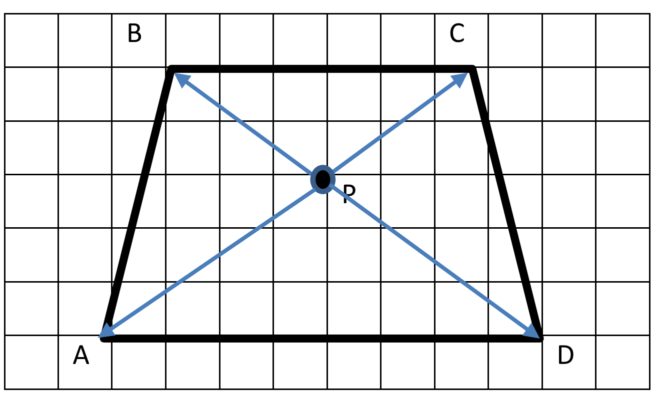


Step 1: Measure the distances from point P to each vertex of the diamond.
The distances from point P to the vertices are PA, PB, PC, and PD, respectively.
Step 2: Multiply these distance by the scale factor $k = 2$.
The distances of the new points from P are: $2 \cdot PA$, $2 \cdot PB$, $2 \cdot PC$, and $2 \cdot PD$.
Step 3: Measure these distances from point P in the direction of each vertex
A, B, C, and D as shown below.



Example 7.3.3: Shrink a Trapezoid by a Factor of ½

Scale the trapezoid by a factor of ½, using P as the center.

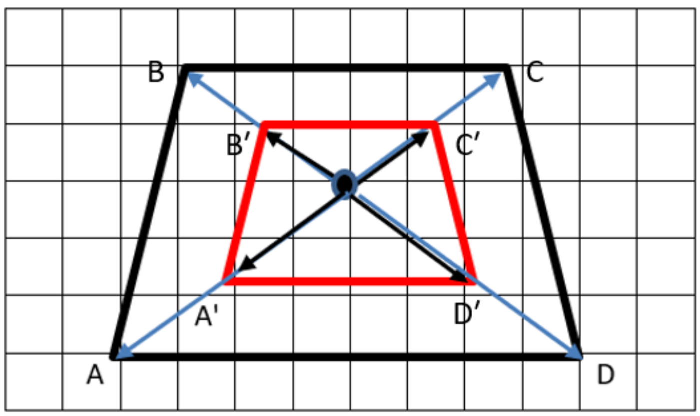


The procedure is the same as the last example.

Step 1: Measure the distances from point P to each vertex of the trapezoid.
The distances from point P to the vertices are PA, PB, PC, and PD, respectively.

Step 2: Multiply these distances by the scale factor $\frac{1}{2}$.
The distances of the new points from P are: $\frac{1}{2}$ PA, $\frac{1}{2}$ PB, $\frac{1}{2}$ PC, and $\frac{1}{2}$ PD.

Step 3: Measure these distances from point P in the direction of each vertex A, B, C, and D as shown below.



Shapes that have been transformed by a scaling transformation (either enlarging or shrinking) are *similar* figures to the original shape.

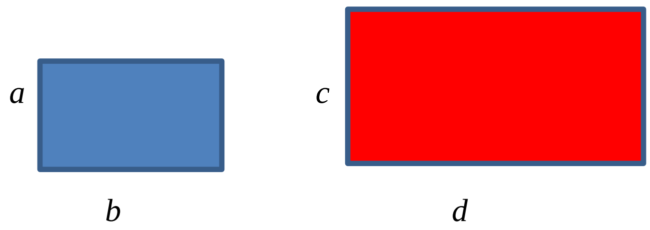
Similarity

Two figures are similar if and only if there exists a combination of an isometry (rigid motion) and a size transformation that generates one figure as the image of the other.

Definition**:** Figures that are the same shape, but not necessarily the same size, are called **similar**. The corresponding angles must be equal and the corresponding sides must be proportional.

Example 7.3.4: Properties of Similar Rectangles

Two rectangles as shown below are similar if and only if the corresponding sides are proportional to each other. In other words, they are similar if $\frac{a}{b} = \frac{c}{d}$.



Suppose that the two rectangles are similar and are related by the scaling factor k. We want to investigate how the perimeters and areas of the rectangles are related.

We know that the sides of the rectangles are related to each other by: $c = ka$ and $d = kb$.

Let P = perimeter of the smaller rectangle and P' = perimeter of the larger rectangle.

⇨ $P' = c + c + d + d = 2c + 2d$.

Substituting $c = ka$ and $d = kb$, we get:

$$P' = 2ka + 2kb = k(2a + 2b).$$

And since $P = 2a + 2b$, we see that $P' = kP$.
Thus, scaling a rectangle up by a factor of k increases the perimeter by a factor of k.

Now let A = area of the smaller rectangle and A' = the area of the larger rectangle.

Then $A = a \cdot b$ and $A' = c \cdot d$. And since $c = ka$ and $d = kb$, we get:

$$A' = c \cdot d = ka \cdot kb = k^2 \cdot a \cdot b = k^2 A$$

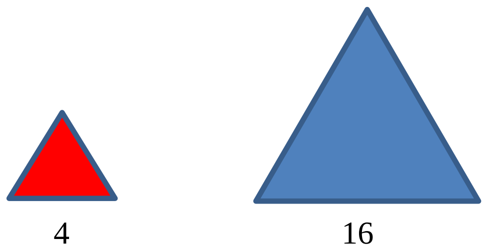
Thus, scaling a rectangle up by a factor of k increases the area by a factor of k^2.

These relationships between perimeters and areas will hold for any closed geometric figure that encloses a region of the plane (e.g. polygons, circles, ovals, etc.)
We summarize these results below:

Theorem: Suppose that figures F and F' are two similar figures where F' is obtained from F by a scaling transformation with scale factor k.

- If P represents the perimeter of F and P' represents the perimeter of F', then the perimeters are related by $P' = kP$.
- If A represents the area of F and A' represents the area of F', then the two areas are related by $A' = k^2A$.

Example 7.3.5: Areas and Perimeters of Similar Triangles



Suppose that the triangles shown above are both equilateral, and hence similar.

a. Determine the perimeter P' of the larger triangle if the perimeter P of the smaller triangle is 12 *mm*.

b. Find the area A' enclosed by the larger triangle if the area enclosed by the smaller triangle is $A = 6.9\ mm^2$.

Solution: First, find the scale factor using the fact that similar triangles have sides that are proportional to each other: $k = \frac{16}{4} = 4$.

a. $P' = kP$

=> $P' = 4 \cdot 12$ mm

=> $P' =$ **48 mm**

b. $A' = k^2A$

=> $A' = 4^2 \cdot 6.9$ mm^2

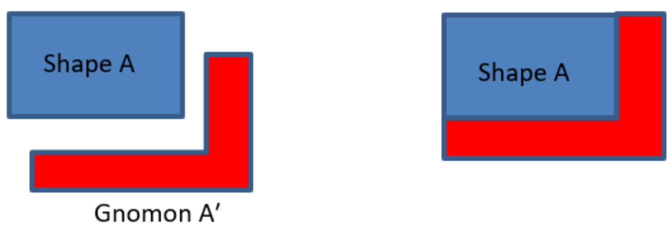
=> $A' =$ **110.4 mm^2**

Gnomons

A **gnomon** is a shape which, when added to a shape A, yields another shape A' that is similar to the original shape A.
Alternately, a **gnomon** is a shape which, when subtracted from a shape A', yields another shape A which is similar to the original shape A'.

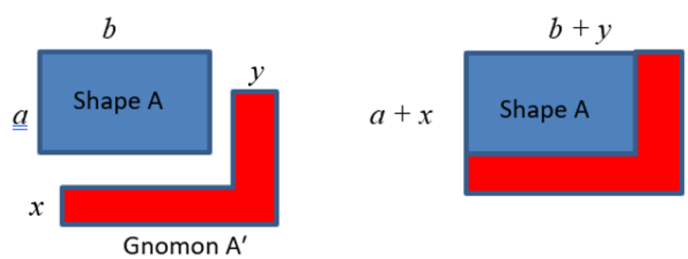
Example 7.3.6: A Gnomon for a rectangle

One general way to get a gnomon for a rectangle is to use an "L-shape" figure.
Let the blue rectangle shown below be shape A. Let the red L-shape be A'.



From the diagram, it appears that the L-shape could be a gnomon for the blue rectangle. One obvious requirement is that the dimensions of the inside of the L-shape must match the height and width of the blue rectangle, since then attaching the L-shape to the blue rectangle forms a new larger rectangle.

In order for A' to be a gnomon to A, the blue rectangle must also be similar to the combined blue and red rectangle. And that means that the corresponding side lengths must be proportional. So we assign side lengths as shown:



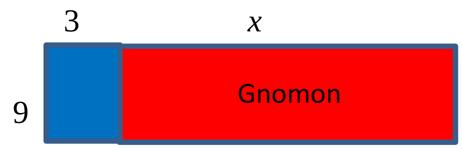
Then set the ratios of corresponding sides equal to one another:

$$\frac{a}{b} = \frac{a+x}{b+y}$$

There is another way to have a gnomon for a rectangle. Given a rectangle, we can attach another rectangle to the longer edge.

Example 7.3.7: Rectangular Gnomon

Find the value of x so that the red larger rectangle on the right is a gnomon to the blue smaller rectangle on the left.



Solution: In order for the red rectangle to be a gnomon to the blue rectangle, the small blue rectangle and combined red and blue rectangle must be similar. I.e. the side lengths must be proportion to one another.

To calculate the value x, set up an equation so that the ratio of corresponding side lengths of the small blue rectangle on the left is equal to the ratio of side lengths of the combined blue and red rectangles on the right. We'll set up the proportion so that the ratio on each side is the width divided by the length. Specifically, we have the width over the length of the small blue rectangle on the left and the width over the length of the combined rectangles on the right:

$$\frac{3}{9} = \frac{9}{3 + x}$$

Cross-multiply to get: $3(3 + x) = 81$

Distribute to get: $9 + 3x = 81$

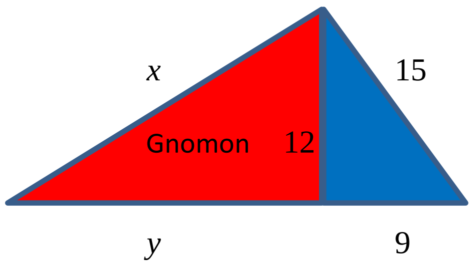
Subtract 9: $3x = 72$

Divide by 3: $x = 24$

Thus, if $\boldsymbol{x = 24}$, then the red larger rectangle is a gnomon to the blue smaller rectangle. And the gnomon (red rectangle) has width 9 and length 24.

Example 7.3.8: Triangular Gnomon

Find the values of x and y so that the larger red triangle on the left is a gnomon to the smaller blue triangle on the right.



Solution: To calculate these values, set up two separate proportions, one with x and one with y, so that the sides of the smaller blue triangle on the right are proportional to the sides of the blue and red triangles combined. To set up the proportion for x, make ratios of the longer leg over the shorter leg of the smaller blue triangle on the right and the longer leg over the shorter leg of the combined triangles.

$$\frac{9}{12} = \frac{15}{x}$$
$$9x = 180$$
$$x = 20$$

Next, set up the proportion for y. Make ratios of the shorter leg over the hypotenuse of the smaller blue triangle on the right and the shorter leg over the hypotenuse of the combined triangles.

$$\frac{9}{15} = \frac{15}{y+9}$$
$$9(y+9) = 225$$
$$9y + 81 = 225$$
$$9y = 144$$
$$y = 16$$

Thus if $\boldsymbol{x = 20}$ and $\boldsymbol{y = 16}$, then the red larger triangle on the left is a gnomon to the blue smaller triangle on the right.

Section 7.4: Fibonacci Numbers and the Golden Ratio

A famous and important sequence is the Fibonacci sequence, named after the Italian mathematician known as Leonardo Pisano, whose nickname was Fibonacci, and who lived from 1170 to 1230. This sequence is:

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ldots\ldots..\}$$

The Fibonacci sequence is defined **recursively.** This means each term is calculated by a formula involving previous terms in the sequence. More precisely:

> The **Fibonacci sequence** is defined by the recursive formula
>
> $$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3$$
>
> where $f_1 = 1$ and $f_2 = 1$.

In other words, to get the next term in the sequence, add the two previous terms.

Let's calculate a few terms. Using $f_1 = 1$ and $f_2 = 1$, and the formula, we have:

$f_3 = f_2 + f_1 = 1 + 1 = 2$
$f_4 = f_3 + f_2 = 2 + 1 = 3$
$f_5 = f_4 + f_3 = 3 + 2 = 5$
$f_6 = f_5 + f_4 = 5 + 3 = 8$
$f_7 = f_6 + f_5 = 8 + 5 = 13$

and so on.

Example 7.4.1: Finding Fibonacci Numbers Recursively

Given that 13th and 14th Fibonacci numbers are $f_{13} = 233$ and $f_{14} = 377$, respectively, use the recursive definition above to find the 15th Fibonacci number.

Solution: Substitute $n = 15$ into the formula $f_n = f_{n-1} + f_{n-2}$ to get:

$$f_{15} = f_{14} + f_{15}\ .$$

Then substitute the given values of f_{14} and f_{13} to get:

$$f_{15} = f_{14} + f_{15} = 233 + 377 = \mathbf{610}$$

Calculating terms of the Fibonacci sequence can be tedious when using the recursive formula, especially when finding terms with a large n. Luckily, a mathematician named Leonhard Euler discovered a formula for calculating any Fibonacci number. This formula was lost for about 100 years and was rediscovered by another mathematician named Jacques Binet. This is now known as Binet's formula, and is shown below.

Binet's Formula: The n^{th} Fibonacci number is given by the following formula:

$$f_n = \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}{\sqrt{5}}$$

Binet's formula is an example of an **explicitly** defined sequence. This means that terms of the sequence are not dependent on previous terms.

A somewhat more user-friendly, simplified version of Binet's formula is sometimes used instead of the one above.

Binet's Simplified Formula: The n^{th} Fibonacci number is obtained by calculating

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and rounding to the nearest integer.

To understand why this formula also produces the Fibonacci number f_n, we look at the second exponential term in Binet's formula:

$$\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

The base of this exponential expression is $\frac{1-\sqrt{5}}{2} \approx -.618$. Since this base has magnitude less than 1, raising it to powers of n produces numbers that quickly approach 0.
This is illustrated by the table:

n	$\left(\frac{1-\sqrt{5}}{2}\right)^n$
1	$-.61803$
2	$.38197$
5	$-.09017$
10	$.00813$
15	$-.000733$
20	$.000061$
25	$-.000006$
30	$.0000005$
35	$-.00000005$

Thus, as n grows, the second term exponential term quickly vanishes, and so

$$f_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Moreover, from the table, we can see that the error is always less than .5 (much less!) and so rounding to the nearest whole number yields the exact value of f_n.

The next few examples illustrate how this simplified formula is used.

Example 7.4.2: Finding f_{10} Explicitly

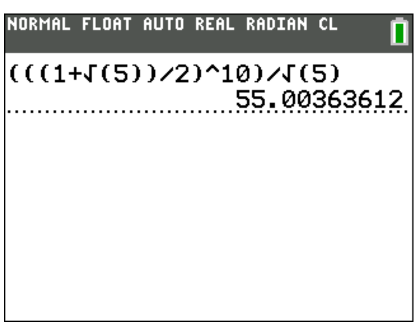
Calculate f_{10} using Binet's simplified formula:

Solution:

$$f_{10} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} \approx \frac{1.618034^{10}}{\sqrt{5}} = 55.00364$$

Rounding, we get $f_{10} = 55$.

Here is the calculator work for f_{10}, using the exact value for $\left(\frac{1+\sqrt{5}}{2}\right)$:



Example 7.4.3: Finding f_{19} Explicitly

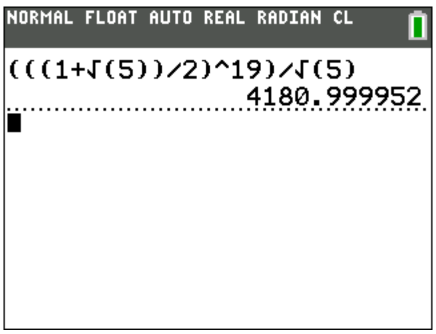
Find the value of f_{19} using Binet's simplified formula.

Solution:

$$f_{19} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{19} \approx \frac{1.618034^{19}}{\sqrt{5}} = 4181.000504$$

Rounding, we get $f_{19} = 4181$.

Here is the calculator work for f_{19}, using the exact value for $\left(\frac{1+\sqrt{5}}{2}\right)$.



Example 7.4.4: Finding f_{40} Explicitly

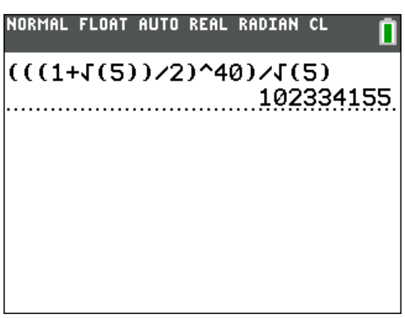
Find the value of f_{40} using Binet's simplified formula.

Solution:

$$f_{40} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{40} = 102{,}334{,}155$$

Rounding, we get $f_{40} = 102{,}334{,}155$.

Here is the calculator work for f_{40}:



We can observe many examples of Fibonacci numbers in nature. For example, the number of branches on certain trees or the number of petals of certain flowers are often Fibonacci numbers.

Example 7.4.5: Fibonacci Numbers and Daisies

a. Daisy with 13 petals

b. Daisy with 21 petals

a. 

b. 

(Daisies, n.d.)

Fibonacci numbers also appear in spiral growth patterns such as the number of spirals on a cactus or in sunflowers seed beds.

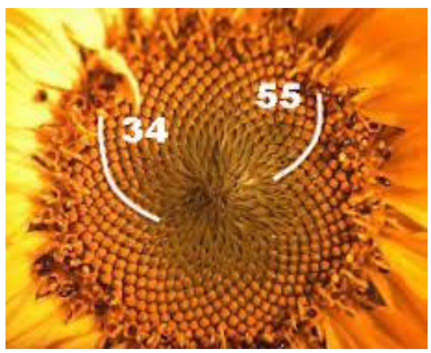
Example 7.4.6: Fibonacci Numbers and Spiral Growth

a. Cactus with 13 clockwise spirals

b. Sunflower with 34 clockwise spirals and 55 counterclockwise spirals

a. 

(Cactus, n.d.)

b. 

(Sunflower, n.d.)

Another interesting pattern arises when looking at the ratios of consecutive Fibonacci numbers.

$$\frac{1}{1}=1,\quad \frac{2}{1}=2,\quad \frac{3}{2}=1.5,\quad \frac{5}{3}=1.\overline{66},\quad \frac{8}{5}=1.6,\quad \frac{13}{8}=1.625,\quad \frac{21}{13}=1.615385,$$
$$\frac{34}{21}=1.619048,\quad \frac{55}{34}=1.617647,\quad \frac{89}{55}=1.618182,\quad \frac{144}{89}=1.617978,\quad \frac{233}{144}=1.618056,$$
$$\frac{377}{233}=1.618026,\quad \frac{610}{377}=1.618037,\quad \frac{987}{610}=1.618033,\ \ldots$$

These ratios appear to be approaching a fixed number; we recognize this number as

$$\left(\frac{1+\sqrt{5}}{2}\right)^n \approx 1.618$$

which we saw in Binet's formula. And we can easily explain this limiting value of these ratios by the simplified version of Binet's formula. Since

$$f_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n,$$

a little bit of algebra shows that

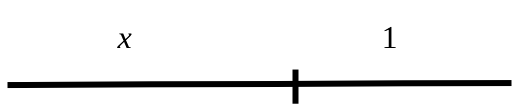
$$\frac{f_{n+1}}{f_n} \approx \frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}} = \left(\frac{1+\sqrt{5}}{2}\right)$$

The value that these ratios are approaching is a special number called the *Golden Ratio* which is denoted by ϕ (the Greek letter phi).

The Golden Ratio is the number $\phi = \dfrac{1+\sqrt{5}}{2}$

The Golden Ratio has the decimal approximation of $\phi \approx 1.61803399$.

This number was studied extensively by the ancient Greeks, and it was thought to have a special significance. They called this the *divine proportion* and it was used in both art and architecture. It is claimed by some to be the most pleasing ratio to the eye. To find this ratio, the Greeks cut a length into two parts, and let the smaller piece equal one unit. The most pleasing cut is when the ratio of the whole length $x+1$ to the long piece x is the same as the ratio of the long piece x to the short piece 1.



This gives the ratio: $\frac{x+1}{x}=\frac{x}{1}$.

Cross-multiplying, we get: $x + 1 = x^2$

Next, we rearrange get the equation: $x^2 - x - 1 = 0$.

Now, solve this quadratic equation using the quadratic formula:

$$a=1,\ b=-1,\ \text{and}\ c=-1$$
$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$$
$$x=\frac{1+\sqrt{5}}{2}\ \text{and}\ x=\frac{1-\sqrt{5}}{2}$$

The Golden Ratio is the positive solution to the quadratic equation $x^2 - x - 1 = 0$, meaning it has the property $\phi^2 = \phi + 1$. This means that if you want to square the Golden Ratio, just add one to it. We can also verify this by plugging in ϕ:

$$\phi^2=\phi+1$$
$$\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{1+\sqrt{5}}{2}+1$$

Doing the calculation, you can see that both sides (rounded to three places) are equal to 2.618.

Another interesting relationship between the Golden Ratio and the Fibonacci sequence occurs when taking powers of ϕ:

$$\phi^3=\phi\cdot\phi^2=\phi(\phi+1)=\phi^2+\phi=(\phi+1)+\phi=2\phi+1$$
$$\phi^3=2\phi+1$$
$$\phi^4=\phi\cdot\phi^3=\phi(2\phi+1)=2\phi^2+\phi=2(\phi+1)+\phi=3\phi+2$$
$$\phi^4=3\phi+2$$
$$\phi^5=\phi\cdot\phi^4=\phi(3\phi+2)=3\phi^2+2\phi=3(\phi+1)+2\phi=5\phi+3$$
$$\phi^5=5\phi+3$$
$$\phi^6=\phi\cdot\phi^5=\phi(5\phi+3)=5\phi^2+3\phi=5(\phi+1)+3\phi=8\phi+5$$
$$\phi^6=8\phi+5$$

Notice that the coefficients on ϕ and the numbers added to the ϕ term are consecutive Fibonacci numbers. This can be generalized to a formula known as the Golden Power Rule:

> **Golden Power Rule:** $\phi^n = f_n\phi + f_{n-1}$
>
> where f_n is the nth Fibonacci number and ϕ is the Golden Ratio.

Example 7.4.7: Powers of the Golden Ratio

Find the following using the golden power rule: a. ϕ^8 and b. ϕ^{15}.

a. $\phi^n = f_n\phi + f_{n-1}$

$\phi^8 = f_8\phi + f_{8-1}$

$\phi^8 = f_8\phi + f_7$

$\phi^8 = 21(1.618) + 13 = 46.978$

b. $\phi^n = f_n\phi + f_{n-1}$

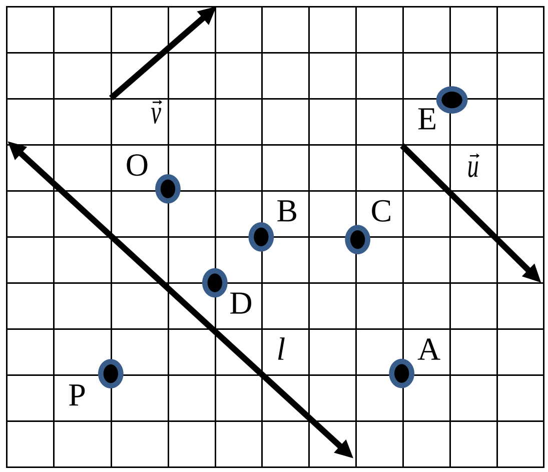
$\phi^{15} = f_{15}\phi + f_{15-1}$

$\phi^{15} = f_{15}\phi + f_{14}$

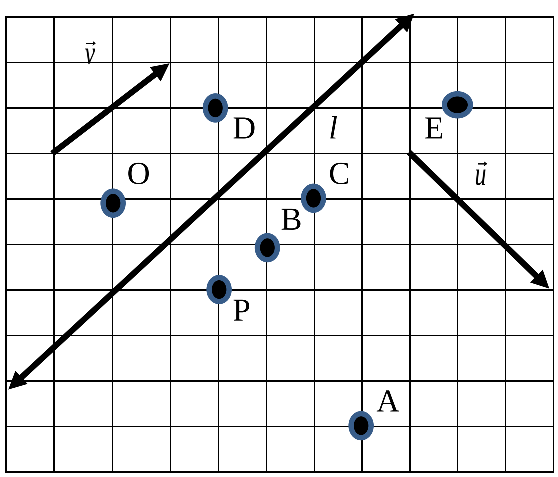
$\phi^{15} = 610(1.618) + 377 = 1363.98$

Chapter 7 Homework

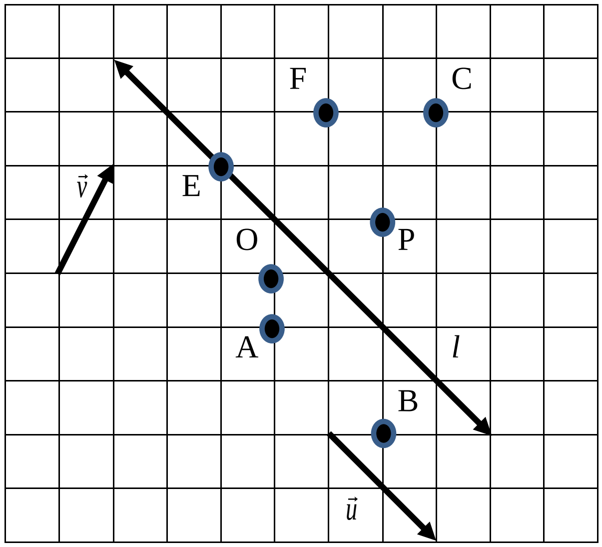
1. Identify the image of point P under the following transformations.
    a. a translation along vector $\vec{v}$
    b. a reflection across line *l*
    c. a counterclockwise rotation of 90° about point O
    d. A glide-reflection across *l* and along $\vec{u}$



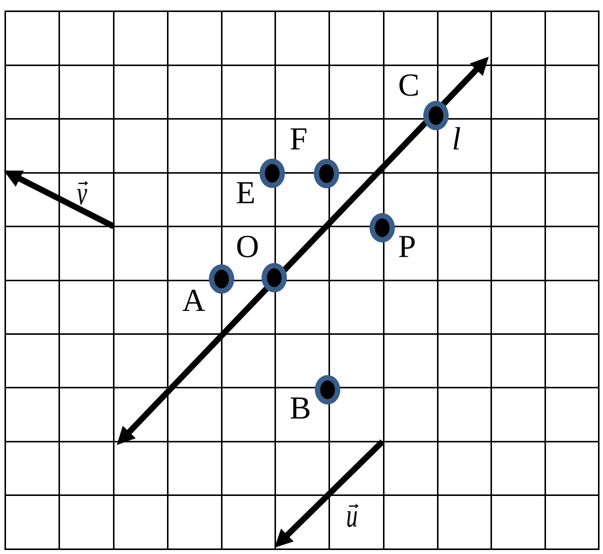
2. Identify the image of point P under the following transformations.
    a. a translation along vector $\vec{u}$
    b. a reflection across line *l*
    c. a counterclockwise rotation of 90° about point O
    d. A glide-reflection across *l* and along $\vec{v}$



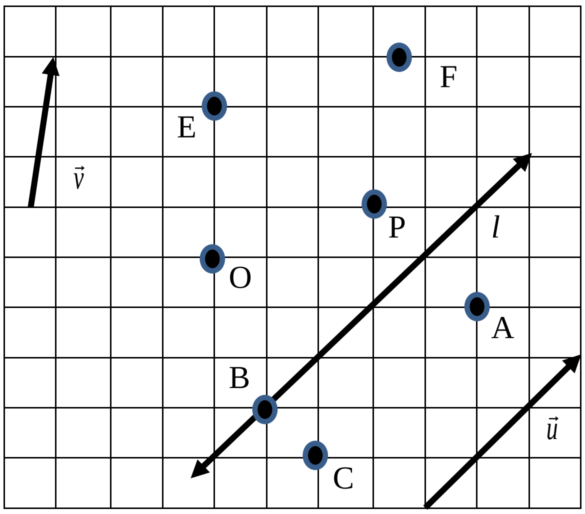
3. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a counterclockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $\vec{u}$



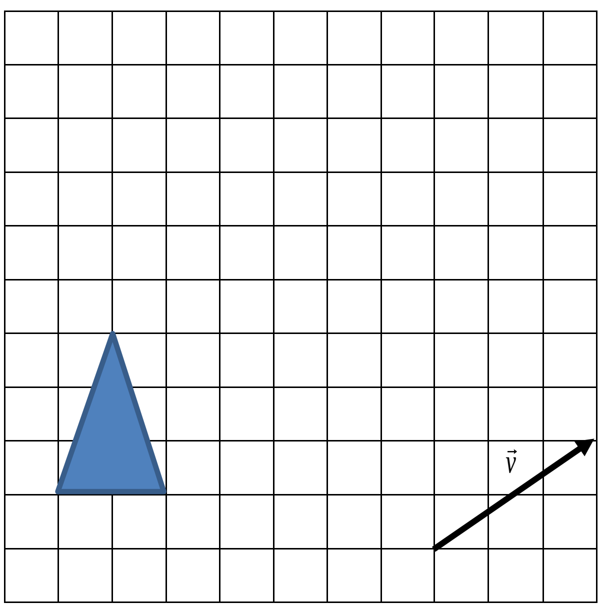
4. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a clockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $\vec{u}$



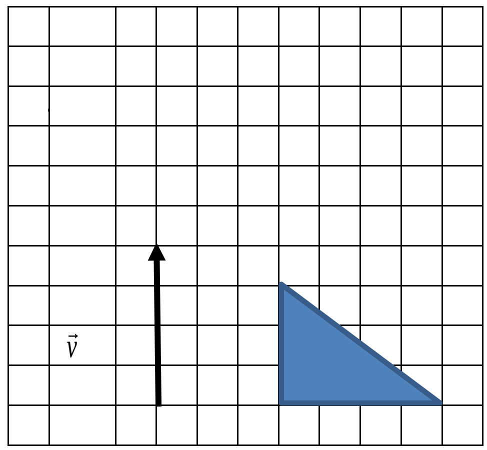
5. Identify the image of point P under the following transformations.
   a. a translation along vector $\vec{v}$
   b. a reflection across line l
   c. a clockwise rotation of $90°$ about point O
   d. A glide-reflection across l and along $-\vec{u}$



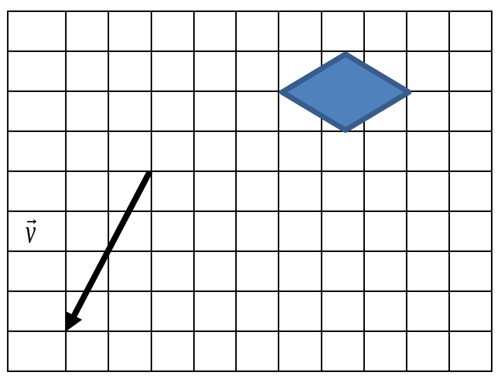
6. Translate the figure along vector $\vec{v}$.



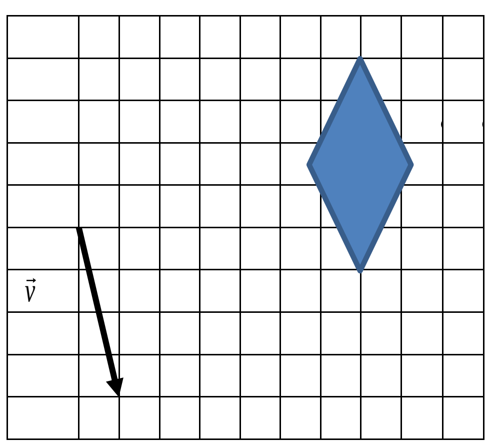
7. Translate the figure along vector $\vec{v}$.



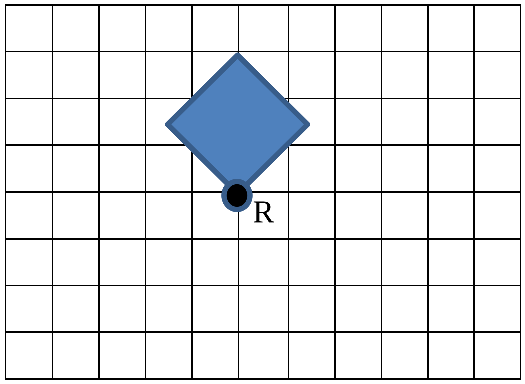
8. Translate the figure along vector $\vec{v}$.



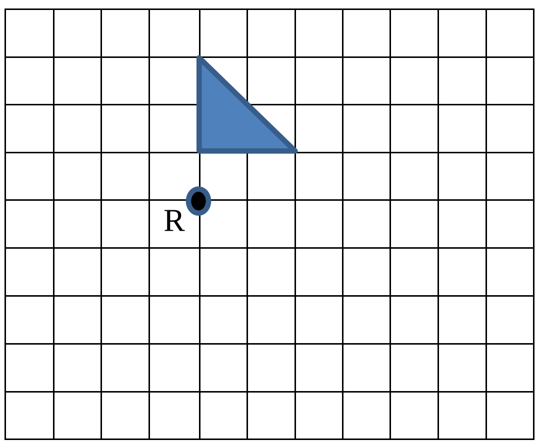
9. Translate the figure along vector $\vec{v}$.



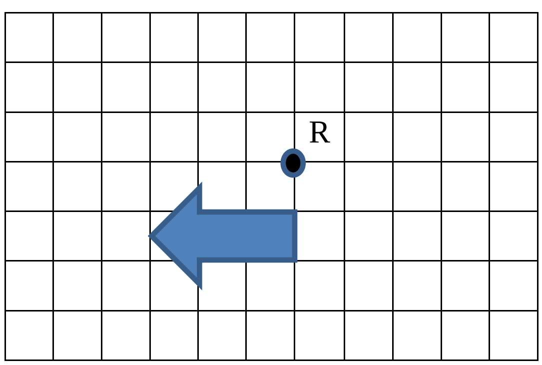
10. Rotate the figure 90° clockwise about the rotocenter R.



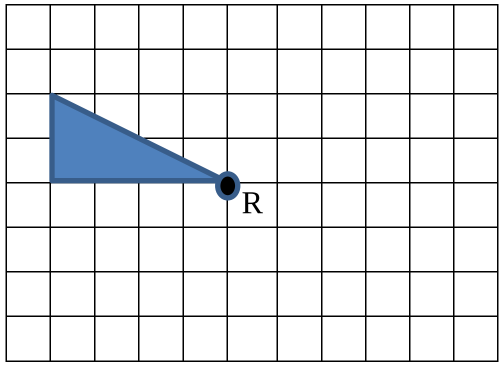
11. Rotate the figure 90° clockwise about the rotocenter R.



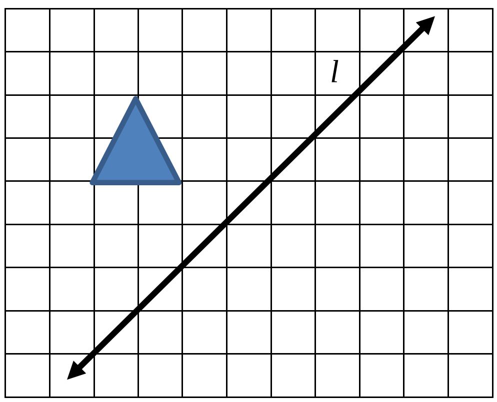
12. Rotate the figure 180° clockwise about the rotocenter R.



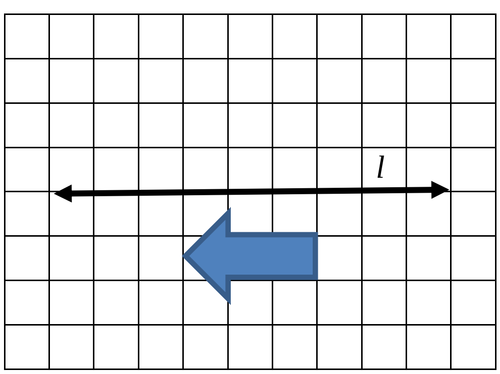
13. Rotate the figure 180° clockwise about the rotocenter R.



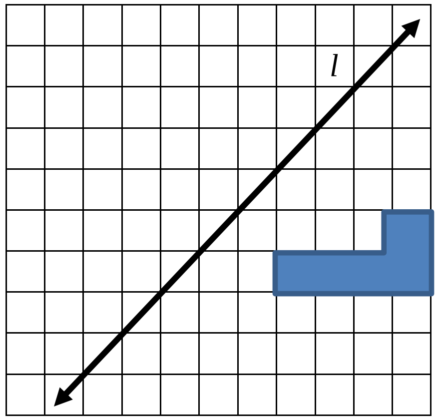
14. Reflect the figure over the line *l*.



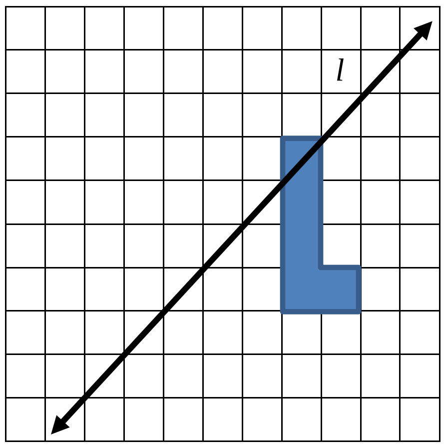
15. Reflect the figure over the line *l*.



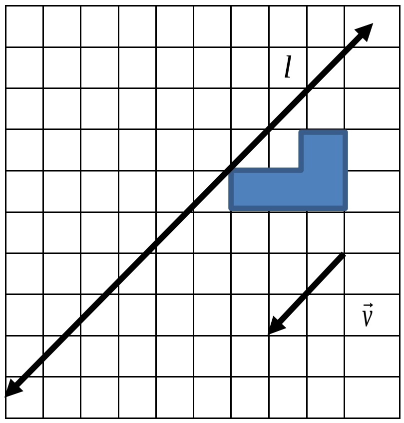
16. Reflect the figure over the line *l*.



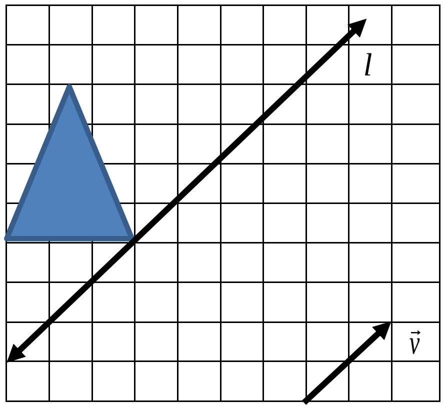
17. Reflect the figure over the line *l*.



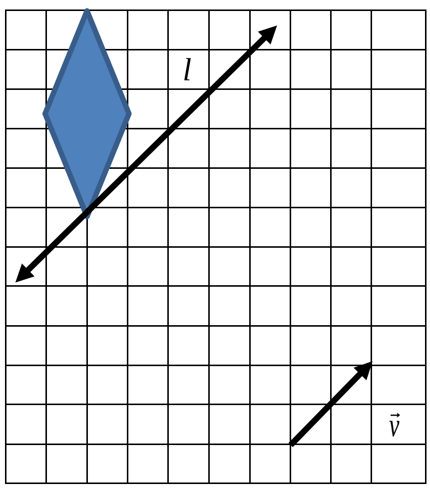
18. Glide-reflect the figure over the line *l* and along the vector $\vec{v}$.



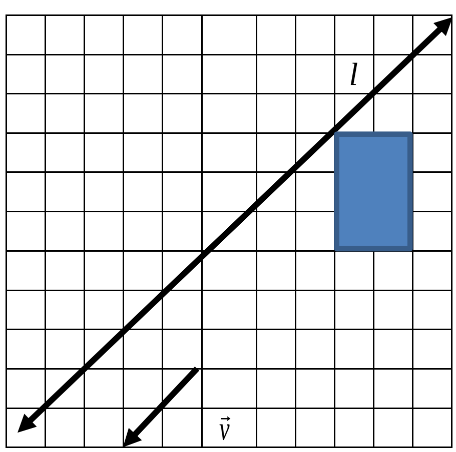
19. Glide-reflect the figure over the line *l* and along the vector $\vec{v}$.



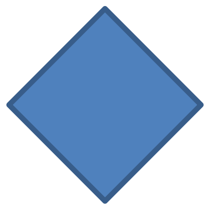
20. Glide-reflect the figure over the line *l* and along the vector $\vec{v}$.



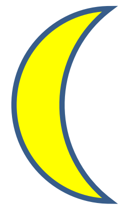
21. Glide-reflect the figure over the line *l* and along the vector $\vec{v}$.



22. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

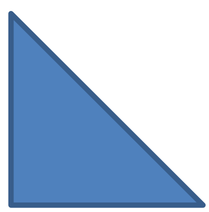


a.

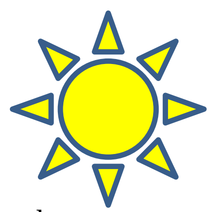


b.

23. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

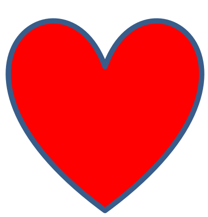


a.

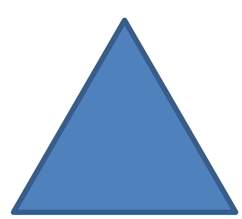


b.

24. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.



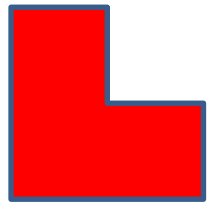
a.



b.

25. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

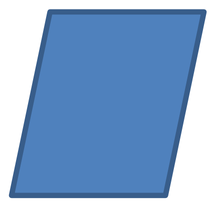
a.



b.

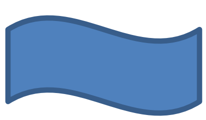
26. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

a.

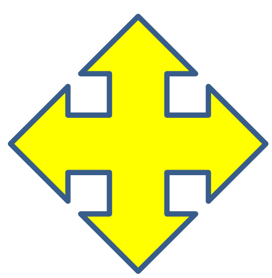


b.

27. In the figures below, identify the types of symmetry. If there are rotation symmetries, identify the degree(s) of rotation. If there are reflection symmetries, draw the line(s) of reflection.

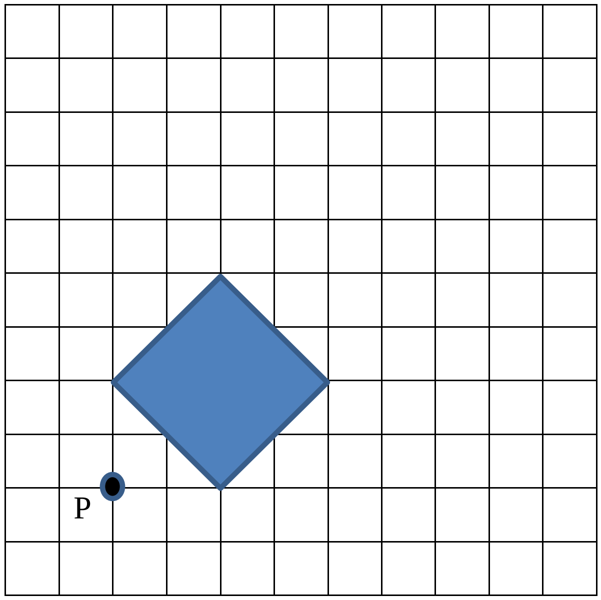


a.

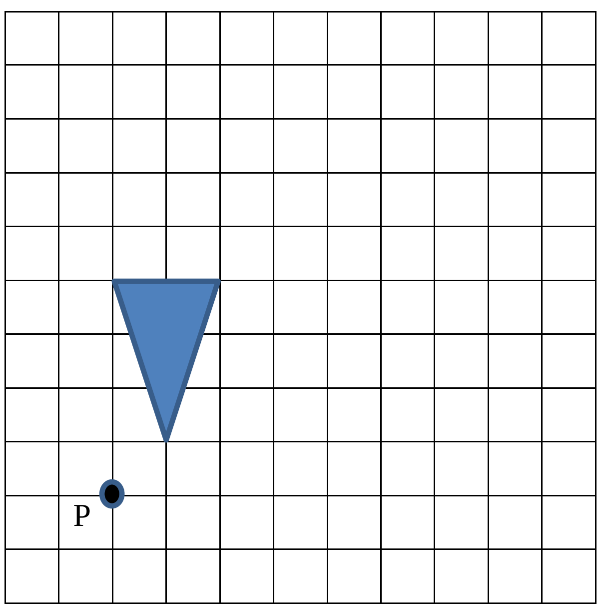


b.

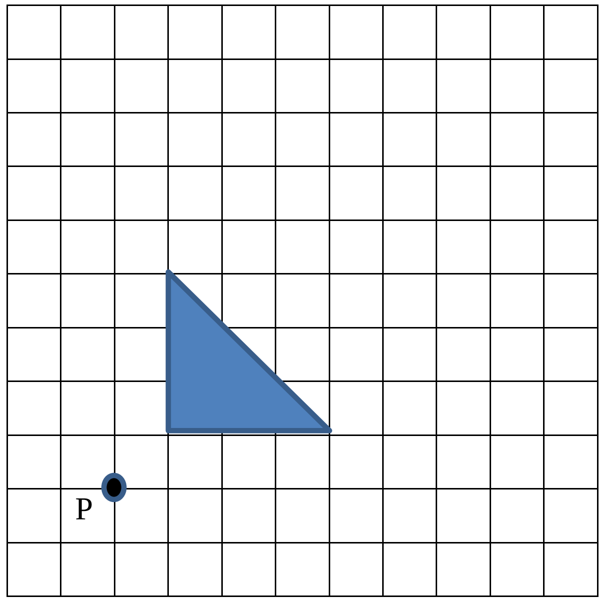
28. Enlarge the figure with respect to the point P by a factor of 2.



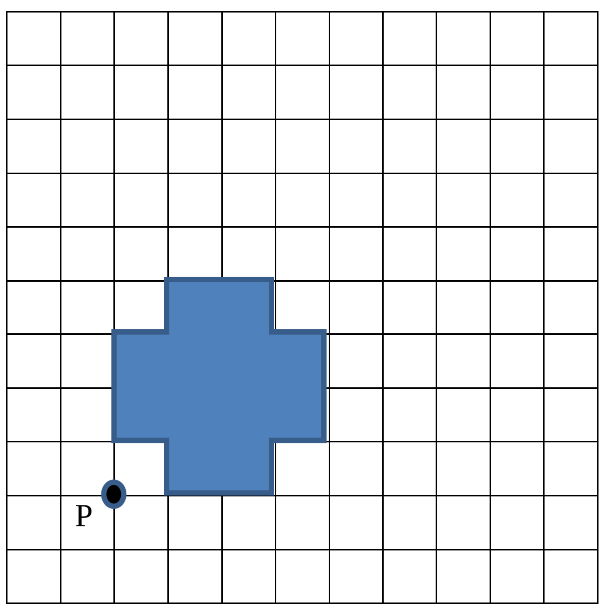
29. Enlarge the figure with respect to the point P by a factor of 2.



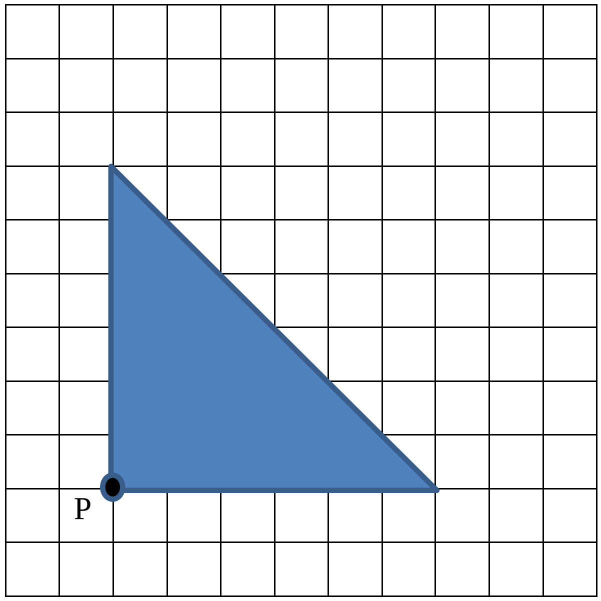
30. Enlarge the figure with respect to the point P by a factor of 2.



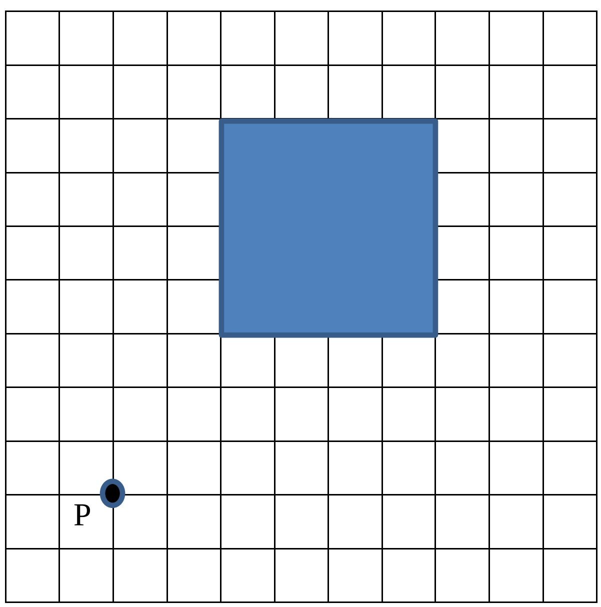
31. Enlarge the figure with respect to the point P by a factor of 2.



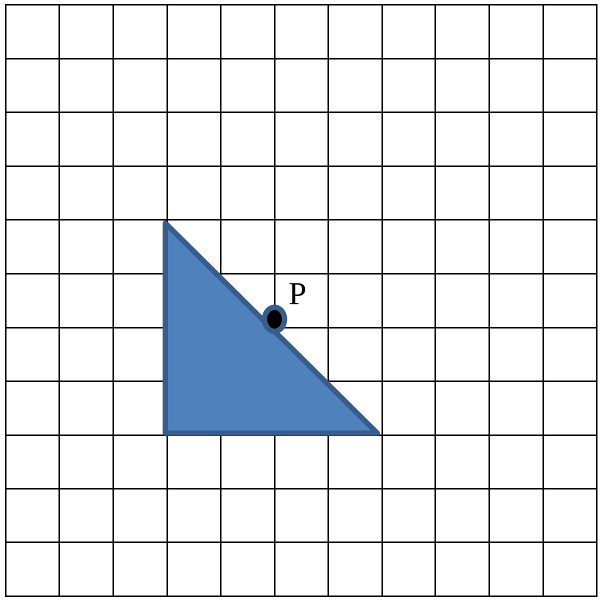
32. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



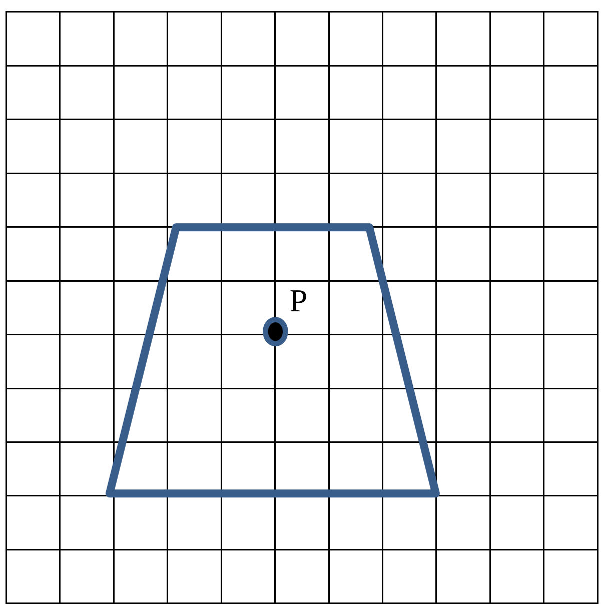
33. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



34. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



35. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



36. Triangles A and A' are similar and are related by the scale factor of 3.

   a. If the perimeter of triangle A is 12 *ft*, find the perimeter of A'.
   b. If the area of triangle A is 8 ft^2, find the area of A'.

37. Triangles A and A' are similar and are related by the scale factor of 4.

   a. If the perimeter of triangle A is 48 *ft*, find the perimeter of A'.
   b. If the area of triangle A is 140 ft^2, find the area of A'.

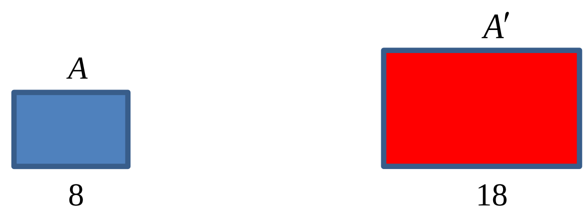
38. Triangles A and A' are similar and are related by the scale factor of 5.

   a. If the perimeter of triangle A is 42 *ft*, find the perimeter of A'.
   b. If the area of triangle A is 68 ft^2, find the area of A'.

39. Triangles A and A' are similar and are related by the scale factor of 2.

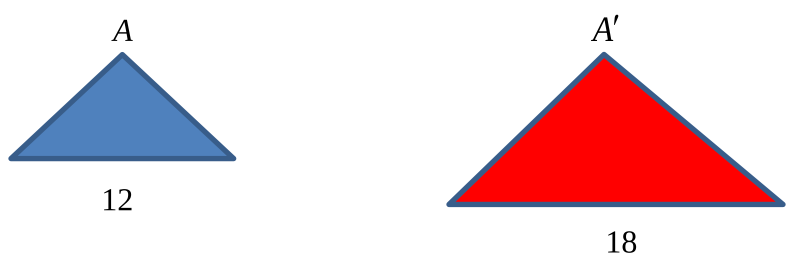
   a. If the perimeter of triangle A is 42 *ft*, find the perimeter of A'.
   b. If the area of triangle A is 80 ft^2, find the area of A'.

40. The shapes A and A' are similar.



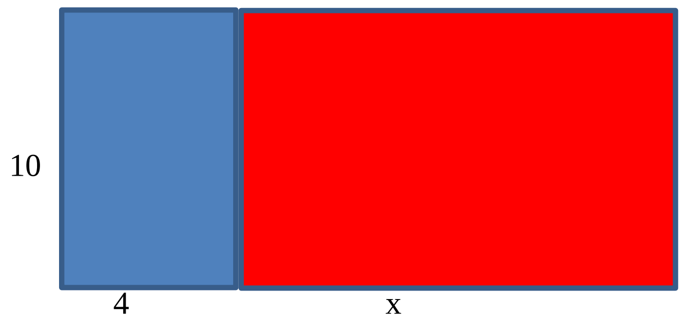
   a. If the perimeter of rectangle A is 22 *ft*, find the perimeter of A'.
   b. If the area of rectangle A is 24 ft^2, find the area of A'.

41. The shapes A and A' are similar.

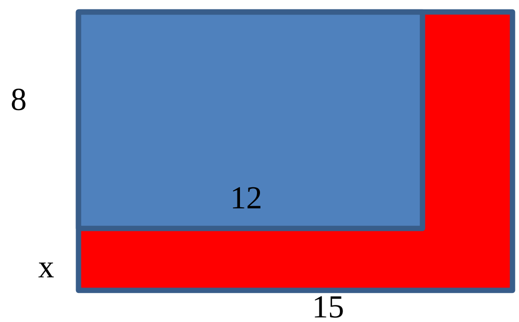


   a. If the perimeter of triangle A is 32 *ft*, find the perimeter of A'.
   b. If the area of triangle A is 48 ft^2, find the area of A'.

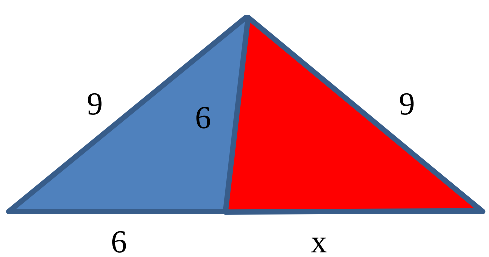
42. Find the value of x so that the red larger rectangle on the right is a gnomon to the blue smaller rectangle on the left.



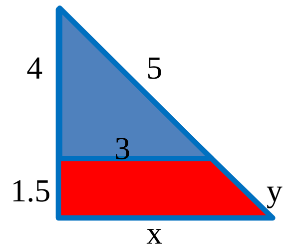
43. Find the value of x so that the red L-shape is a gnomon to the blue rectangle.



44. Find the value of x so that the red triangle on the right is a gnomon to the blue triangle on the left.



45. Find the values of x and y so that the red trapezoid is a gnomon to the blue triangle.



46. Compute the values of the following:
    a. f_{12}
    b. $f_4 + f_3 + f_2$
    c. f_{4+3+2}

47. Compute the values of the following.
    a. f_{13}
    b. $f_5 + f_3 + f_1$
    c. f_{5+3+1}

48. Compute the values of the following using Binet's simplified formula.
    a. f_{54}
    b. f_{54}
    c. f_{8+7+6}

49. Compute the values of the following using Binet's simplified formula.
    a. f_{35}
    b. f_{45}
    c. f_{9+8+7}

50. Given $f_{33} = 3,524,578$ and $f_{34} = 5,702,887$, find f_{32} and f_{35}.

51. Given $f_{40} = 102,334,155$ and $f_{41} = 165,580,141$, find f_{39} and f_{42}.

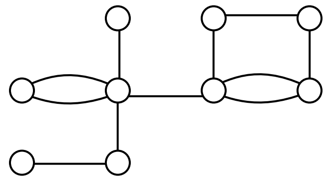
Solutions

to

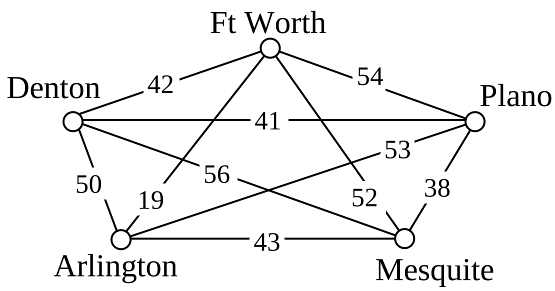
Selected Problems

Chapter 1: Graph Theory

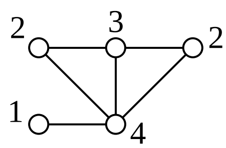
1.



3.



5.

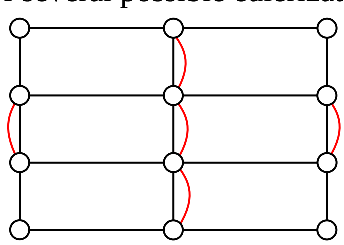


7. The first and the third graphs are connected

9. Bern to Frankfurt to Munchen to Berlin: 12hrs 50 min. (Though trip through Lyon, Paris and Amsterdam only adds 30 minutes)

11. The first graph has an Euler circuit. The last two graphs each have two vertices with odd degree.

13. One of several possible eulerizations requiring 5 duplications:

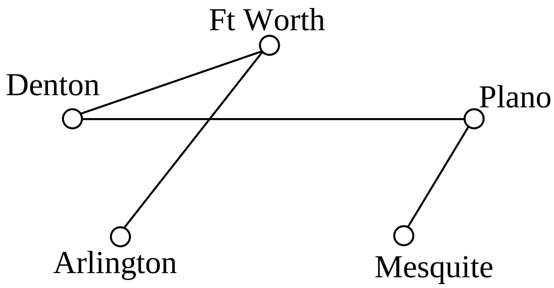


17. Only the middle graph has a Hamiltonian circuit.

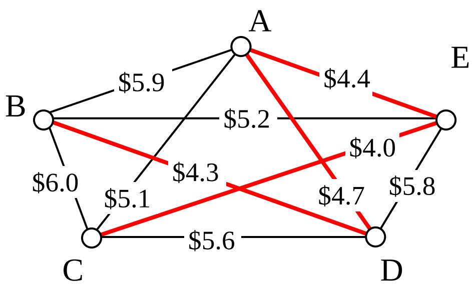
19. a. Ft Worth, Arlington, Mesquite, Plano, Denton, Ft Worth: 183 miles
    b. Same as part a
    c. Same as part a

21. a. ABDCEA
    b. ACEBDA
    c. ADBCEA

23.

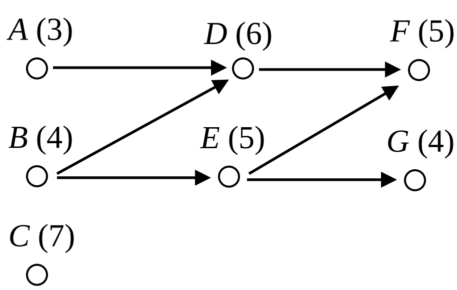


25.

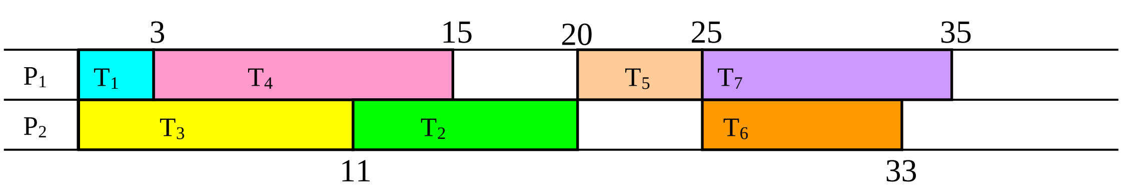


Chapter 2: Scheduling

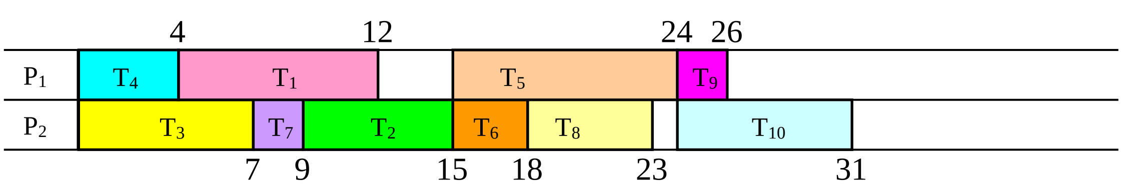
1.



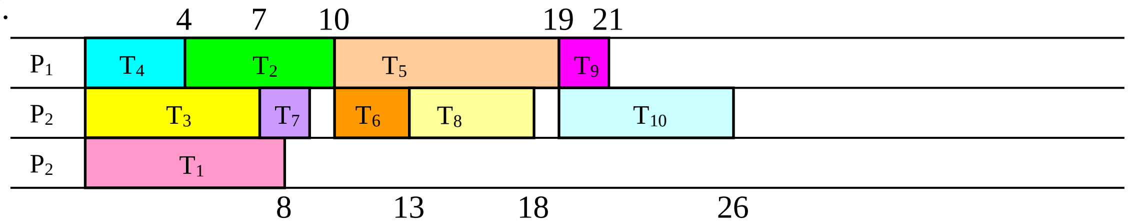
3.



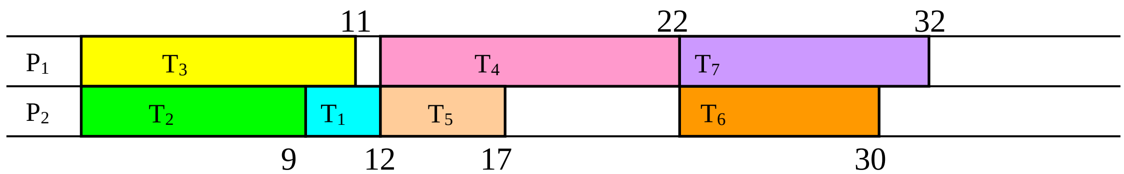
5.



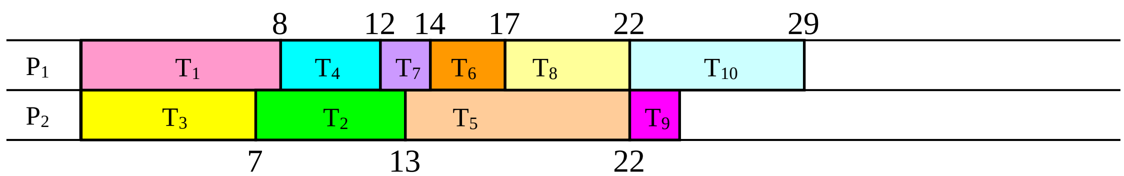
7.



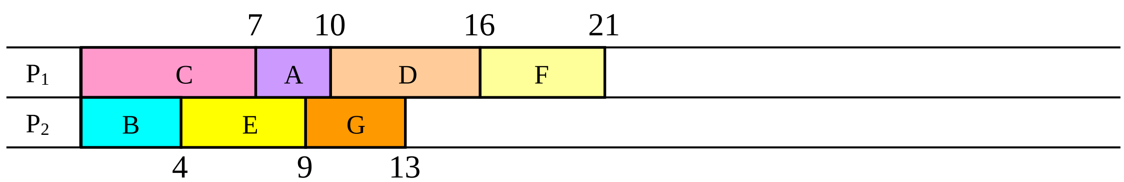
9. Priority List: T_4, T_3, T_7, T_2, T_6, T_5, T_1



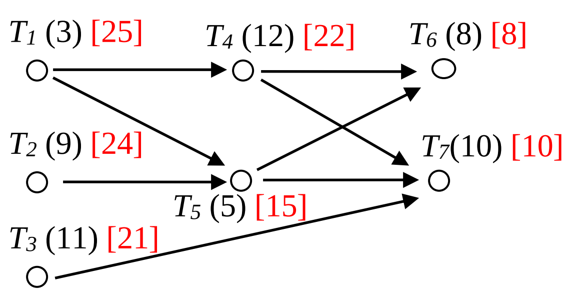
11. Priority List: T_5, T_1, T_3, T_{10}, T_2, T_8, T_4, T_6, T_7, T_9



13. Priority List: C, D, E, F, B, G, A

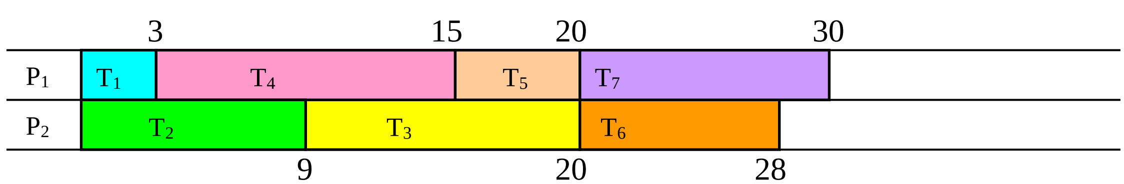


15. a.

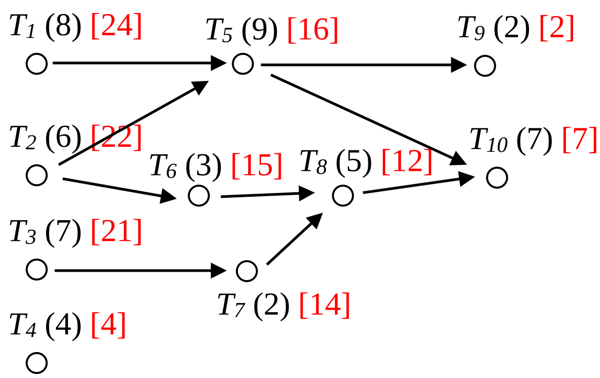


b. Critical path: T_1, T_4, T_7. Minimum completion time: 25

c. Critical path priority list: T_1, T_2, T_4, T_3, T_5, T_7, T_6



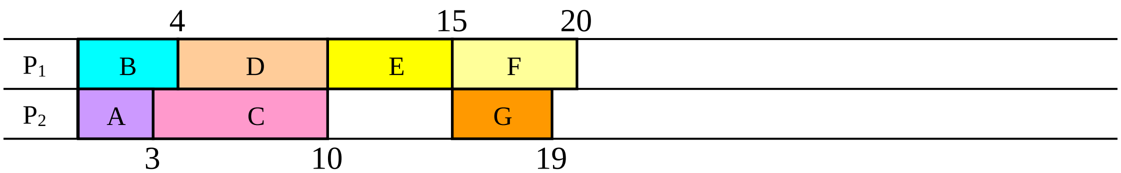
17. a.



b. Critical path: T_1, T_5, T_{10}. Minimum completion time: 24

c. Critical path priority list: T_1, T_2, T_3, T_5, T_6, T_7, T_8, T_{10}, T_4, T_9

19. Critical path priority list: B, A, D, E, C, F, G



Chapter 3: Probability

1. a. $\frac{6}{13}$ b,. $\frac{2}{13}$

3. $\frac{150}{335}$ = 44.8%

5. $\frac{1}{6}$

7. $\frac{26}{65}$

9. $\frac{3}{6}=\frac{1}{2}$

11. $\frac{4}{52}=\frac{1}{13}$

13. $1-\frac{1}{12}=\frac{11}{12}$

15. $1-\frac{25}{65}=\frac{40}{65}$

17. $\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$

19. $\frac{1}{6}\cdot\frac{3}{6}=\frac{3}{36}=\frac{1}{12}$

21. $\frac{17}{49}\cdot\frac{16}{48}=\frac{17}{49}\cdot\frac{1}{3}=\frac{17}{147}$

23. a. $\frac{4}{52} \cdot \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$

b. $\frac{4}{52} \cdot \frac{48}{52} = \frac{192}{2704} = \frac{12}{169}$

c. $\frac{48}{52} \cdot \frac{48}{52} = \frac{2304}{2704} = \frac{144}{169}$

d. $\frac{13}{52} \cdot \frac{13}{52} = \frac{169}{2704} = \frac{1}{16}$

e. $\frac{48}{52} \cdot \frac{39}{52} = \frac{1872}{2704} = \frac{117}{169}$

25. $\frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652}$

27. a. $\frac{11}{25} \cdot \frac{14}{24} = \frac{154}{600}$

b. $\frac{14}{25} \cdot \frac{11}{24} = \frac{154}{600}$

c. $\frac{11}{25} \cdot \frac{10}{24} = \frac{110}{600}$

d. $\frac{14}{25} \cdot \frac{13}{24} = \frac{182}{600}$

e. no males = two females. Same as part d.

29. P(F and A) = $\frac{10}{65}$

31. P(red or odd) = $\frac{6}{14} + \frac{7}{14} - \frac{3}{14} = \frac{10}{14}$. Or 6 red and 4 odd-numbered blue marbles is 10 out of 14.

33. P(F or B) = $\frac{26}{65} + \frac{22}{65} - \frac{4}{65} = \frac{44}{65}$. Or P(F or B) = $\frac{18+4+10+12}{65} = \frac{44}{65}$

35. P(King of Hearts or Queen) = $\frac{1}{52} + \frac{4}{52} = \frac{5}{52}$

37. a. P(even | red) = $\frac{2}{5}$ b. P(even | red) = $\frac{2}{6}$

39. P(Heads on second | Tails on first) = $\frac{1}{2}$. They are independent events.

41. P(speak French | female) = $\frac{3}{14}$

43. Out of 4,000 people, 10 would have the disease. Out of those 10, 9 would test positive, while 1 would falsely test negative. Out of the 3990 uninfected people, 399 would falsely test positive, while 3591 would test negative.

a. P(virus | positive) = $\frac{9}{9+399} = \frac{9}{408}$ = 2.2%

b. P(no virus | negative) = $\frac{3591}{3591+1} = \frac{3591}{3592}$ = 99.97%

45. Out of 100,000 people, 300 would have the disease. Of those, 18 would falsely test negative, while 282 would test positive. Of the 99,700 without the disease, 3,988 would falsely test positive and the other 95,712 would test negative.

P(disease | positive) = $\frac{282}{282+3988} = \frac{282}{4270}$ = 6.6%

47. Out of 100,000 women, 800 would have breast cancer. Out of those, 80 would falsely test negative, while 720 would test positive. Of the 99,200 without cancer, 6,944 would falsely test positive.

P(cancer | positive) = $\frac{720}{720+6944} = \frac{720}{7664}$ = 9.4%

49. $2 \cdot 3 \cdot 8 \cdot 2$ = 96 outfits

51. a. 4 · 4 · 4 = 64 b. 4 · 3 · 2 = 24

53. 26 · 26 · 26 · 10 · 10 · 10 = 17,576,000

55. ${}_4P_4$ or 4 · 3 · 2 · 1 = 24 possible orders

57. Order matters. ${}_7P_4$ = 840 possible teams

59. Order matters. ${}_{12}P_5$ = 95,040 possible themes

61. Order does not matter. ${}_{12}C_4$ = 495

63. ${}_{50}C_6$ = 15,890,700

65. ${}_{27}C_{11}$ · 16 = 208,606,320

67. There is only 1 way to arrange 5 CD's in alphabetical order. The probability that the CD's are in alphabetical order is one divided by the total number of ways to arrange 5 CD's. Since alphabetical order is only one of all the possible orderings you can either use permutations, or simply use 5!. P(alphabetical) = 1/5! = 1/(5 P 5) = $\frac{1}{120}$.

69. There are ${}_{48}C_6$ total tickets. To match 5 of the 6, a player would need to choose 5 of those 6, ${}_6C_5$, and one of the 42 non-winning numbers, ${}_{42}C_1$. $\dfrac{6\cdot 42}{12271512} = \dfrac{252}{12271512}$

71. All possible hands is ${}_{52}C_5$. Hands will all hearts is ${}_{13}C_5$. $\dfrac{1287}{2598960}$.

73. $\$3\left(\dfrac{3}{37}\right)+\$2\left(\dfrac{6}{37}\right)+(-\$1)\left(\dfrac{28}{37}\right)=-\$\dfrac{7}{37}$ = -$0.19

75. There are ${}_{23}C_6$ = 100,947 possible tickets.
Expected value = $\$29{,}999\left(\dfrac{1}{100947}\right)+(-\$1)\left(\dfrac{100946}{100947}\right)$ = -$0.70

77. $\$48(0.993)+(-\$\,302)(0.007)$ = $45.55

Chapter 4: Sets

1. {m, i, s, p}

3. One possibility is: Multiples of 3 between 1 and 10

5. Yes

7. True

9. True

11. False

13. $A \cup B$ = {1, 2, 3, 4, 5}

15. $A \cap C$ = {4}

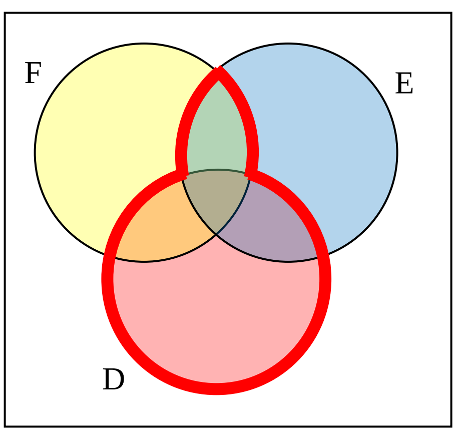
17. A^c = {6, 7, 8, 9, 10}

19. $D^c \cap E$ = {t, s}

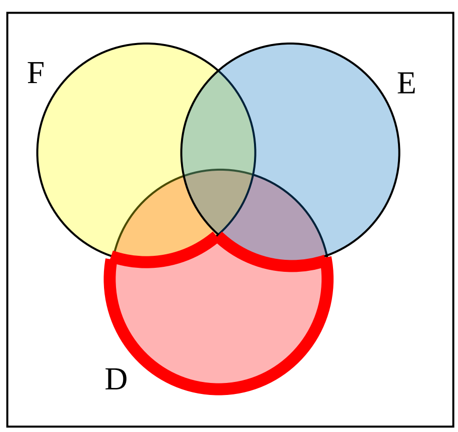
21. $(D \cap E) \cup F$ = {k, b, a, t, h}

23. $(F \cap E)^c \cap D$ = {b, c, k}

25.



27.



29. One possible answer: $(A \cap B) \cup (B \cap C)$

31. $(A \cap B^c) \cup C$

33. 5

35. 6

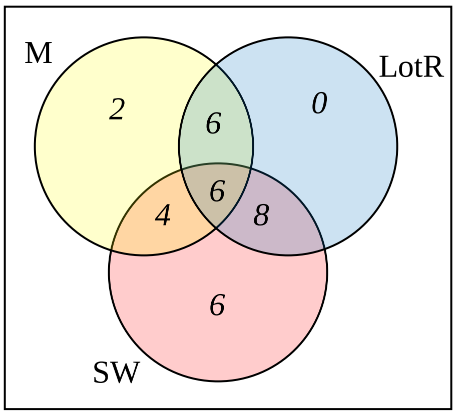
37. $n(A \cap C) = 5$

39. $n(A \cap B \cap C^c) = 3$

41. $n(G \cup H) = 45$

43. 136 use Redbox

45.



a) 8 had seen exactly one

b) 6 had only seen SW

Chapter 5: Logic

1. {5, 15, 25, …}

3. At least one person did not fail the quiz today.

5. a. Elvis is alive or did not gain weight.
   b. It is not the case that Elvis is alive and gained weight.
   c. If Elvis gained weight, then he is not alive.
   d. Elvis is alive if and only if he did not gain weight.

7.

A	B	$\sim A$	$\sim A \vee B$	$\sim(\sim A \vee B)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

9.

A	B	C	$A \lor B$	$\sim C$	$(A \lor B) \rightarrow \sim C$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

11.

A	B	$A \veebar B$
T	T	F
T	F	T
F	T	T
F	F	F

13. The results are identical; the exclusive or translates to "(*A* or *B*) and not (*A* and *B*)".

15. a. Not necessarily true; this is the inverse.
You could get your mouth washed out for some other reason.
b. True; this is the contrapositive.
c. Not necessarily true; this is the converse.
You could get your mouth washed out for some other reason.

17. Luke faces Vader and Obi-Wan interferes.

19. a. This couldn't happen; you fulfilled your part of the bargain but your coach didn't.
b. This couldn't happen; you didn't fulfill your part of the bargain but your coach let you play anyway. This could happen with a conditional statement, but not a biconditional.
c. This could happen; practice = play, no practice = no play.

21. You don't need a dated receipt or you don't need your credit card to return this item.

23. Valid, by the law of contraposition.

25. Valid, by disjunctive syllogism.

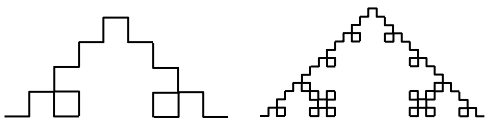
27. Invalid; we are using the inclusive or, so the sets of people with a pencil and people with a pen could possibly overlap. Marcie might be in the intersection of the two sets.

29. False dilemma; you could fly, take a bus, hitchhike…

31. Correlation implies causation; maybe the only time our smoke detector goes off is when I burn dinner, and the kids choose to eat cereal whenever I burn dinner.

Chapter 6: Fractals

1.

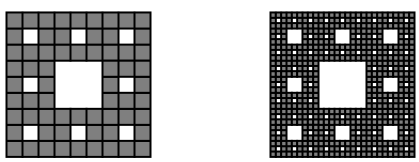


Step 2 Step 3

3.

Step 2 Step 3

5.

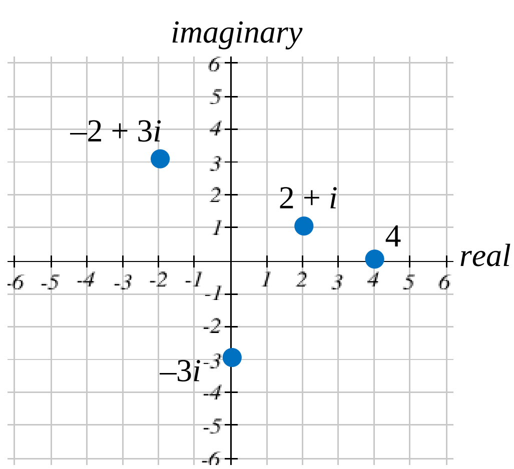


Step 2 Step 3

9. a) $P_1 = (4/3) \cdot 9 = 12$ in. b) $P_4 = (4/3)^4 \cdot 9 = 28.44$ in. c) $A = 12.46$ in^2

11. a) $A_1 = (3/4) \cdot 16 = 12$ in^2 b) $A_4 = (3/4)^4 \cdot 16 = 5.06$ in^2 c) $A = 0$ in^2

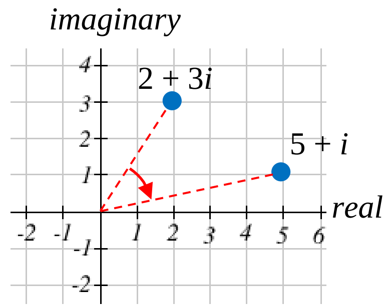
13.



15. a) $5 - i$ b) $5 - 4i$

17. a) $6+12i$ b) $10-2i$ c) $14+2i$

19. $(2+3i)(1-i)=5+i$. It appears that multiplying by 1 – *i* both scaled the number away from the origin, and rotated it clockwise about 45°.



21.
$$z_1 = iz_0 + 1 = i(2) + 1 = 1 + 2i$$
$$z_2 = iz_1 + 1 = i(1+2i) + 1 = i - 2 + 1 = -1 + i$$
$$z_3 = iz_2 + 1 = i(-1+i) + 1 = -i - 1 + 1 = -i$$

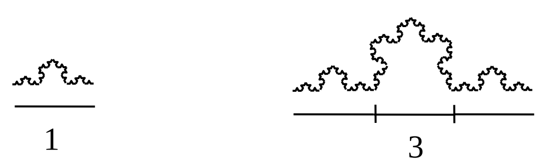
23.
$$z_0 = 0$$
$$z_1 = {z_0}^2 - 0.25 = 0 - 0.25 = -0.25$$
$$z_2 = {z_1}^2 - 0.25 = (-0.25)^2 - 0.25 = -0.1875$$
$$z_3 = {z_2}^2 - 0.25 = (-0.1875)^2 - 0.25 = -0.21484$$
$$z_4 = {z_3}^2 - 0.25 = (-0.21484)^2 - 0.25 = -0.20384$$

25. attracted, to approximately -0.37766 + 0.14242*i*

27. periodic 2-cycle 29. Escaping 31. periodic 3-cycle

33. Four copies of the Koch curve are needed to create a curve scaled by 3.

$$D = \frac{\log(4)}{\log(3)} \approx 1.262$$



35. Eight copies of the shape are needed to make a copy scaled by 3. $D = \dfrac{\log(8)}{\log(3)} \approx 1.893$

37. a) Yes, periodic 3-cycle b) Yes, periodic 3-cycle c) No

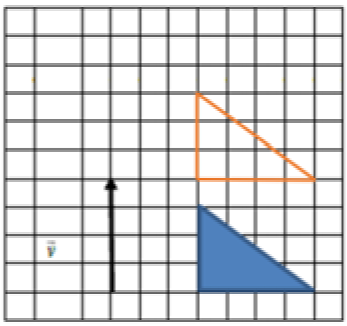
Chapter 7: Symmetry and the Golden Ratio

1. a. D b. B c. C d. A

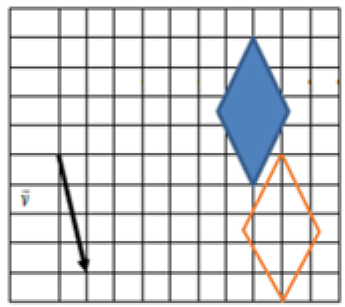
3. a. C b. A c. E d. B

5. a. F b. A c. B d. C

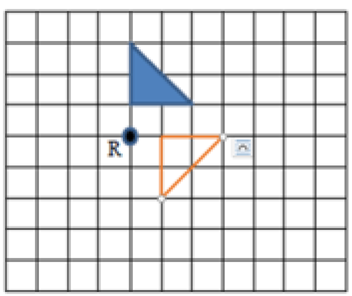
7. Translate the figure along vector $\vec{v}$.



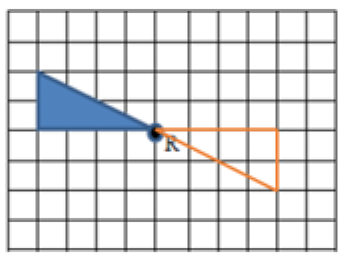
9. Translate the figure along vector $\vec{v}$.



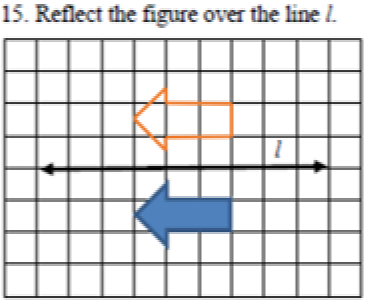
11. Rotate the figure 90° clockwise about the rotocenter R.



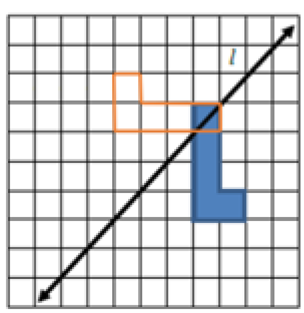
13. Rotate the figure 180° clockwise about the rotocenter R.



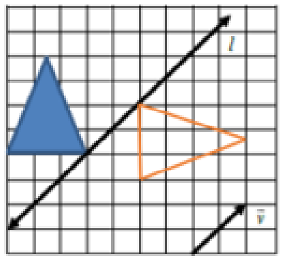
15. Reflect the figure over the line *l*.



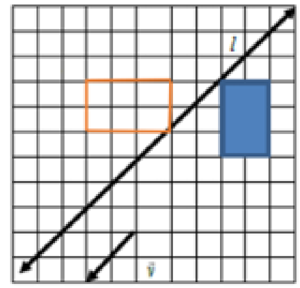
17. Reflect the figure over the line *l*.



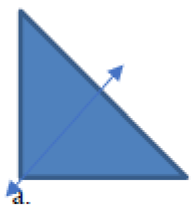
19. Glide-reflect the figure over the line l and along the vector $\vec{v}$.



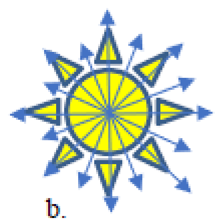
21. Glide-reflect the figure over the line l and along the vector $\vec{v}$.



23.



a.

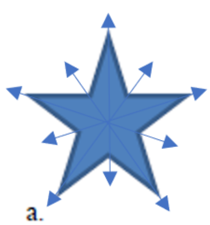


b.

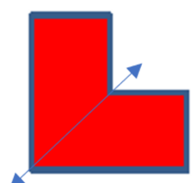
a. D_1 ; 360°

b. D_8 ; 45°,90°,135°,180°,225°,270°,315°,360°

25.



a.



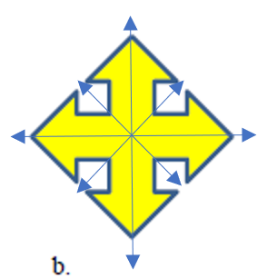
b.

a. D_5 ; 72°,144°,216°,288°,360°

b. D_1 ; 360°

27.

a.

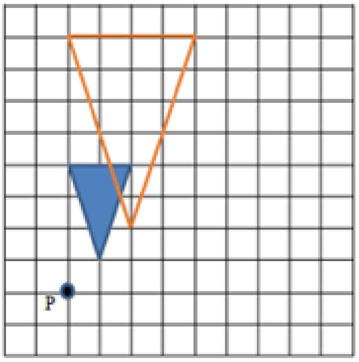


b.

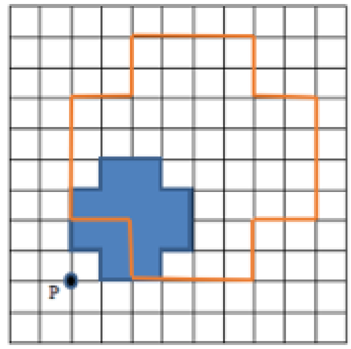
a. Z_2 ; 180°,360°

b. D_4; 90°,180°,270°,360°

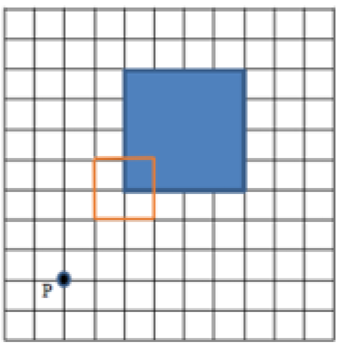
29. Enlarge the figure with respect to the point P by a factor of 2.



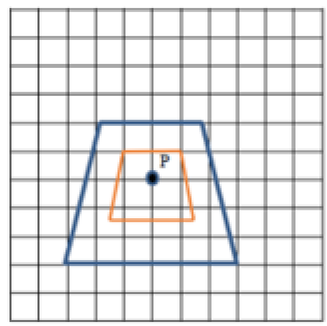
31. Enlarge the figure with respect to the point P by a factor of 2.



33. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



35. Shrink the figure with respect to the point P by a factor of $\frac{1}{2}$.



37. a. P = 192 ft b. A = 2,240 ft^2

39. a. P = 84 ft b. A = 320 ft^2

41. a. P = 48 ft b. A = 108 ft^2

43. $x = 2$

45. $x = 4.125$ $y = 1.875$

47. a. 233 b. 8 c. 34

49. a. 9,227,465 b. 1,134,903,170 c. 46,368

51. f_{39} = 63,245,986 f_{42} = 267,914,296