1. Rational Functions - Review

In this section we will review rational functions and concentrate on how to graph them by hand.

A rational function is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. Even though rational fractions are constructed from polynomials, their graphs look quite different from the graphs of polynomial functions.

Examples of rational functions:
For each function, decide if it is a rational function and then if it is find its domain, x-intercepts and y-intercept.

1. $f(x) = \dfrac{x^2+3}{(x-1)(x+2)}$.

yes

x int : set y = 0 $\dfrac{x^2+3}{(x-1)(x+3)} = 0$

$x^2+3=0 \Rightarrow x^2=-3$ not real number (no x intercepts)

y intercept $f(0) = \dfrac{3}{(-1)(2)} = -3/2$

2. $f(x) = \dfrac{x^2-3}{(x^2+1)(x+2)}$.

yes; x int $\dfrac{x^2-3}{(x^2+1)(x+2)} = 0 \Rightarrow x^2-3=0$ $x^2=3 \Rightarrow x=\pm\sqrt{3}$

x intercepts $(\sqrt{3},0), (-\sqrt{3},0)$

y int : $f(0) = \dfrac{-3}{(1)(2)} = -3/2$

3. $f(x) = \dfrac{x^2+3}{(\sqrt{x}-1)(x+2)}$.

$x^{1/2}$ not a polynomial $f(x)$ is not a rational function.

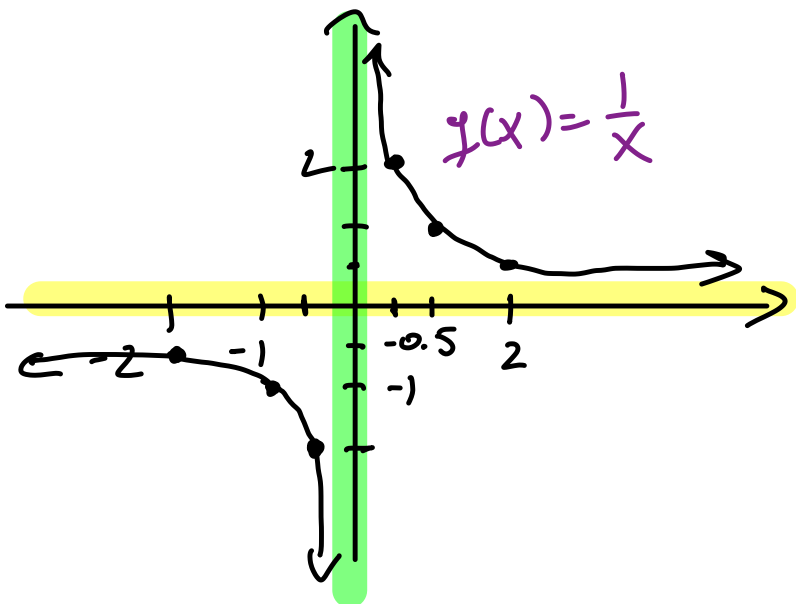
The **domain of a rational function** consists of all real numbers x except those for which the denominator is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those x-values.

2. A simple rational Function $f(x) = \frac{1}{x}$

polynomie

Graph the rational function $f(x) = \frac{1}{x}$ and state its domain and range.

$x \neq 0$ $\quad (-\infty, 0) \cup (0, \infty)$



x	y
$-1/2$	$\frac{1}{-1/2} = -2$
-1	$\frac{1}{-1} = -1$
-2	$\frac{1}{-2} = -0.5$
$1/2$	$\frac{1}{1/2} = 2$
1	$\frac{1}{1} = 1$
2	$\frac{1}{2} = 0.5$

End behaviour: $\quad x \to \infty \quad y \to 0 \qquad x \to -\infty \quad y \to 0$

3. Rational functions and Asymptotes

Begin by factoring the numerator and the denominator. If both the denominator and the numerator have a common factor that reduces, then we say that the rational function has a **hole** at the corresponding x of that factor.

$$f(x) = \frac{\cancel{(x-1)}(x+2)}{(x+1)\cancel{(x-1)}}$$ $f(x)$ has a hole @ $\boxed{x=1}$

Vertical Asymptotes - A rational function has vertical asymptotes where the denominator is zero (after any cancellation of common factors from the denominator and numerator)

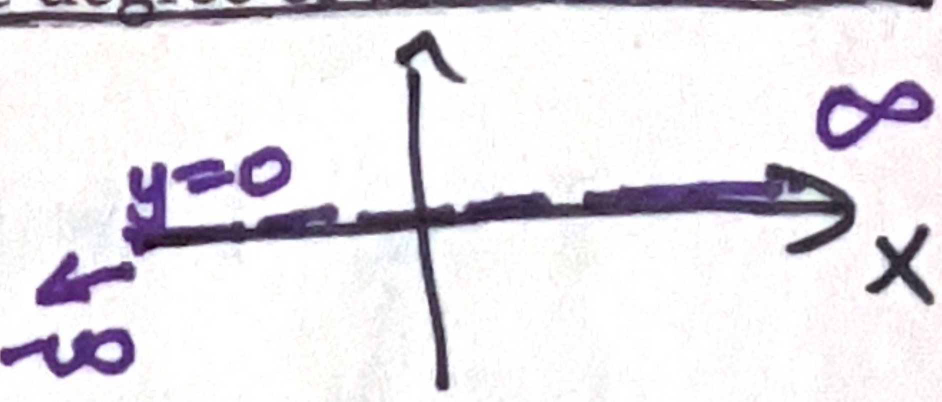
$$f(x) = \frac{x+2}{x+1}$$ $x+1=0$ VA: $x=-1$

never crosses a VA.

Horizontal asymptotes

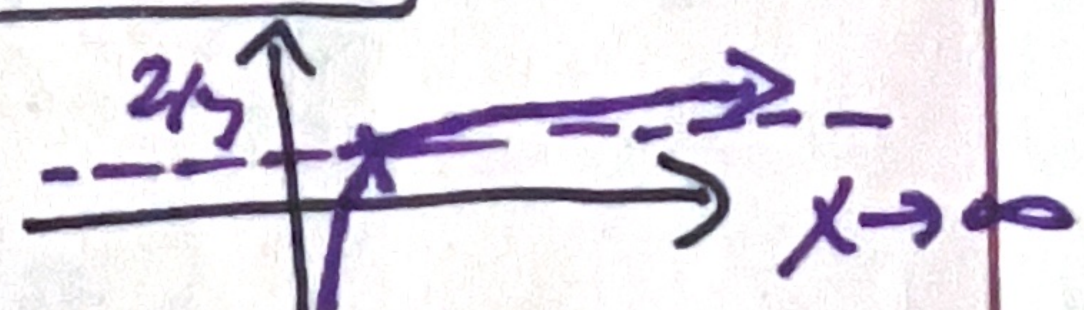
Case 1: If the degree of the numerator is less than the degree of the denominator, the the horizontal asymptote is $y = 0$.

$$f(x) = \frac{x^2+3}{x^4+3x+1}$$ HA: $y=0$



Case 2: If the degree of the numerator equals the degree of the denominator, then the horizontal asymptote is $\boxed{y = \text{the ratio of the leading coefficients.}}$

$$f(x) = \frac{2x^2+3}{3x^2+x+1}$$ HA: $y=\frac{2}{3}$



Case 3: Lastly, if the degree of the numerator is higher than the degree of the denominator, then the rational function has no horizontal asymptotes.

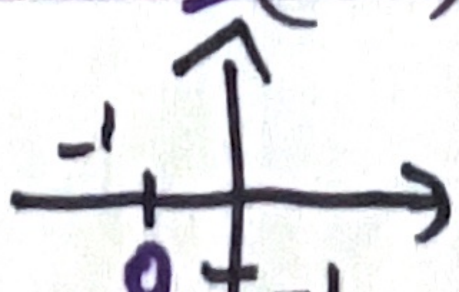
$$f(x) = \frac{x^4+5}{x^2+3}$$ no HA

Find the asymptotes and holes(if any) of the rational function

$$f(x) = \frac{(x^2-1)(2x+3)}{(x+1)(x^2+5x+6)} = \frac{(x-1)\cancel{(x+1)}(2x+3)}{\cancel{(x+1)}(x+2)(x+3)} = \frac{(x-1)(2x+3)}{(x+2)(x+3)}$$

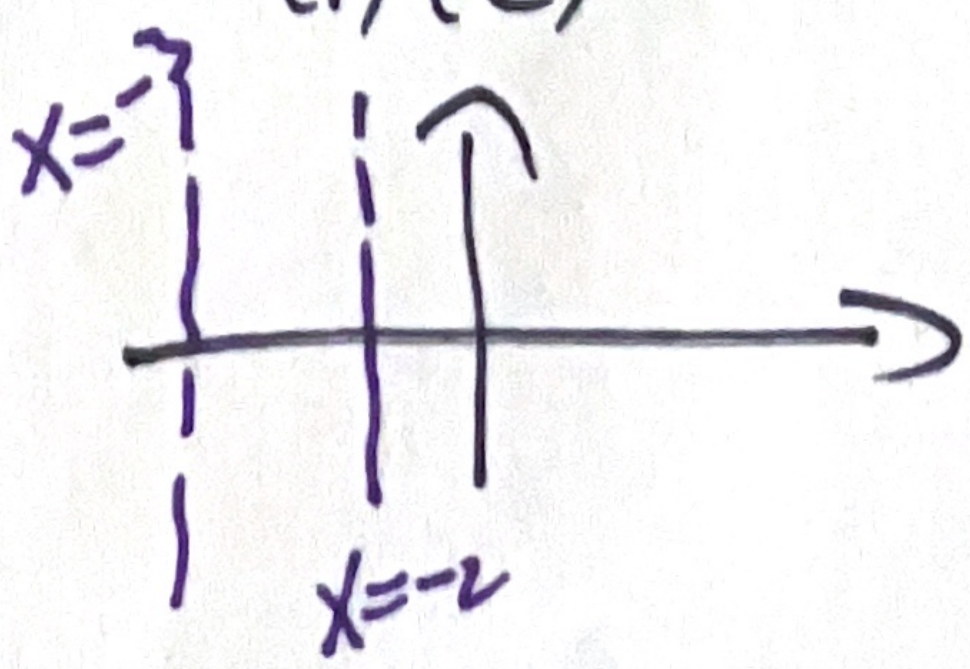
Hole: $x=-1$

$(-1,-1)$

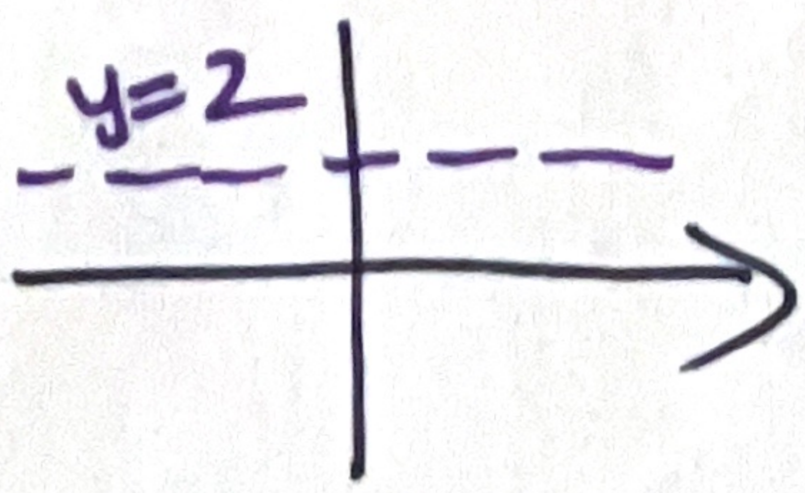


$$f(-1) = \frac{(-2)(1)}{(1)(2)} = -1$$

VA: $x=-2$ & $x=-3$

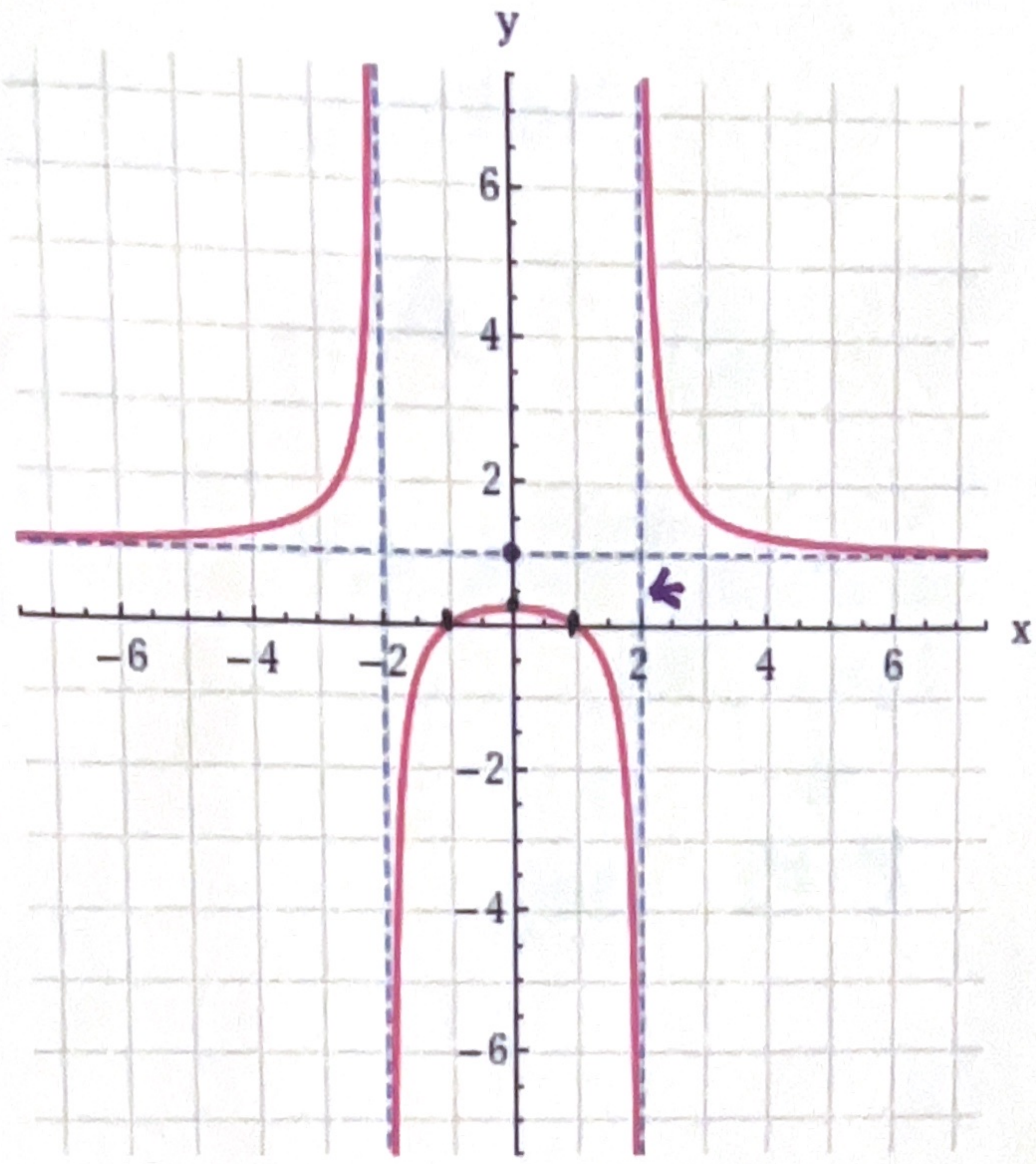


HA: $y=2$



4. Graphs of Rational functions

The graphs of rational functions are more complicated due to the asymptotes. Lets examine the graph of the rational function below.



No Holes

VA: $\begin{cases} x=-2 \\ x=2 \end{cases}$

HA: $y=1$

x-int : $x=-1$
$x=1$

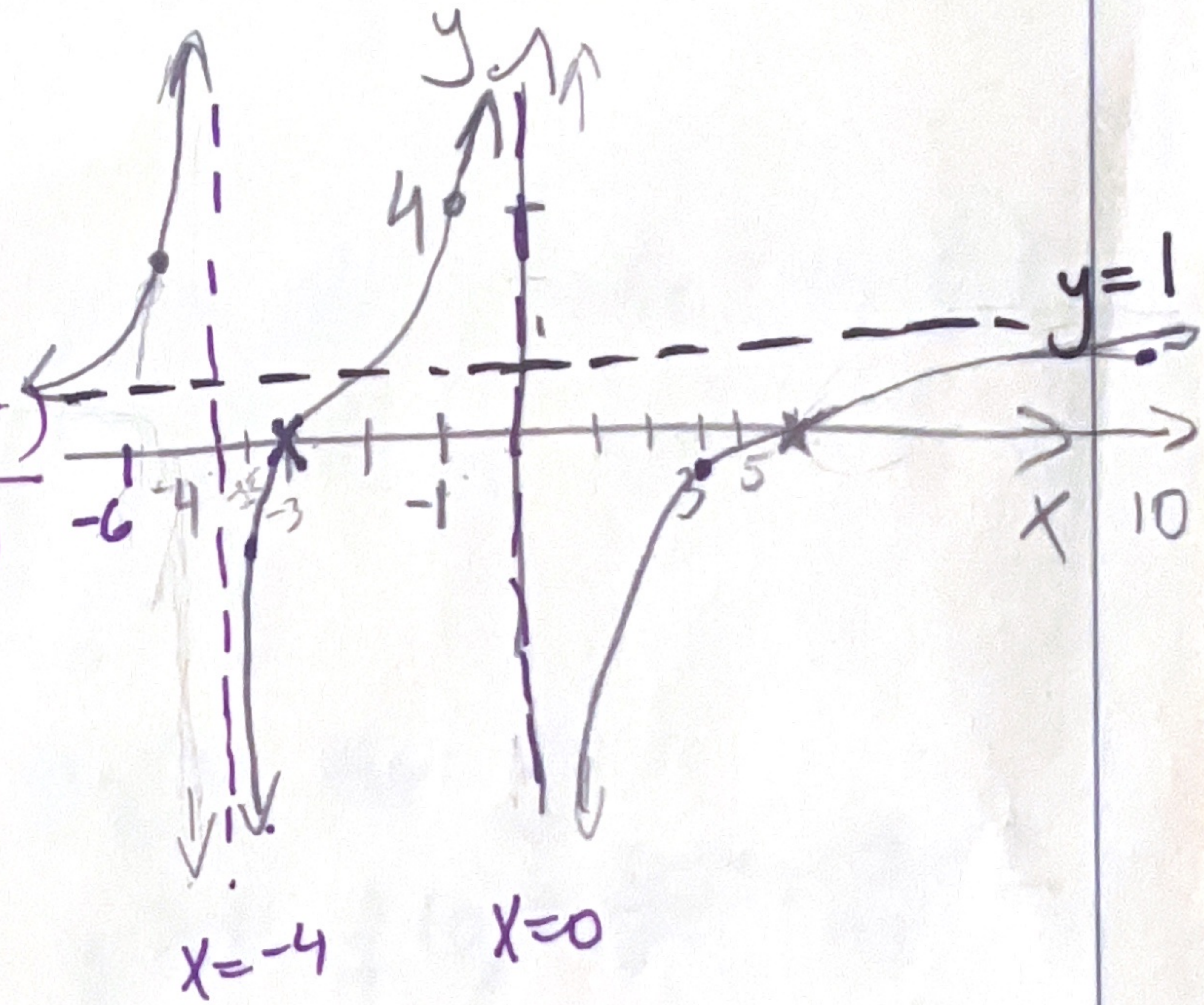
y-int $y=0.5$

$\boxed{\lim_{x\to\infty} f(x) \overset{HA}{=} 1}$ $\boxed{\lim_{x\to-\infty} f(x)=1}$

$\lim_{x\to-2^-} f(x)=\infty$ $\lim_{x\to-2^+} f(x)=-\infty$

How to graph Rational functions

Graph the rational function $f(x) = \frac{x^2 - 2x - 15}{x^2 + 4x} = \frac{(x+3)(x-5)}{x(x+4)}$



Step 0 factor

Step 1 Holes? NO

Step 2 VA: $f(x)=\frac{(x+3)(x-5)}{x(x+4)}$

$x=0$ $x=-4$

Step 3 HA: $y=\frac{1}{1} \Rightarrow \boxed{y=1}$

Step 4 x-int $\frac{(x+3)(x-5)}{x(x+4)} = 0 \Rightarrow x=-3$, $x=5$

y-int $f(0)=?$ No intercept

Step 5 Graph it by graphing each piece.

$f(-6)=\frac{(-3)(-11)}{(-6)(-2)} = \frac{33}{12} = 2.75$

$f(-3.5)=-2.43$

$f(3)=-.57$

Practice: Graph the rational function $f(x) = \frac{x^2-4x-12}{x^2-x-6} = \frac{(x-6)\cancel{(x+2)}}{\cancel{(x+2)}(x-3)}$

Step 0 $f(x) = \frac{x-6}{x-3}$

Step 1 holes: yes $x+2=0$ $\boxed{x=-2}$ hole $(-2, \frac{4}{3})$

$f(-2) = \frac{-2-6}{-2-3} = \frac{-8}{-6} = \frac{4}{3}$

Step 2 VA: $x=3$

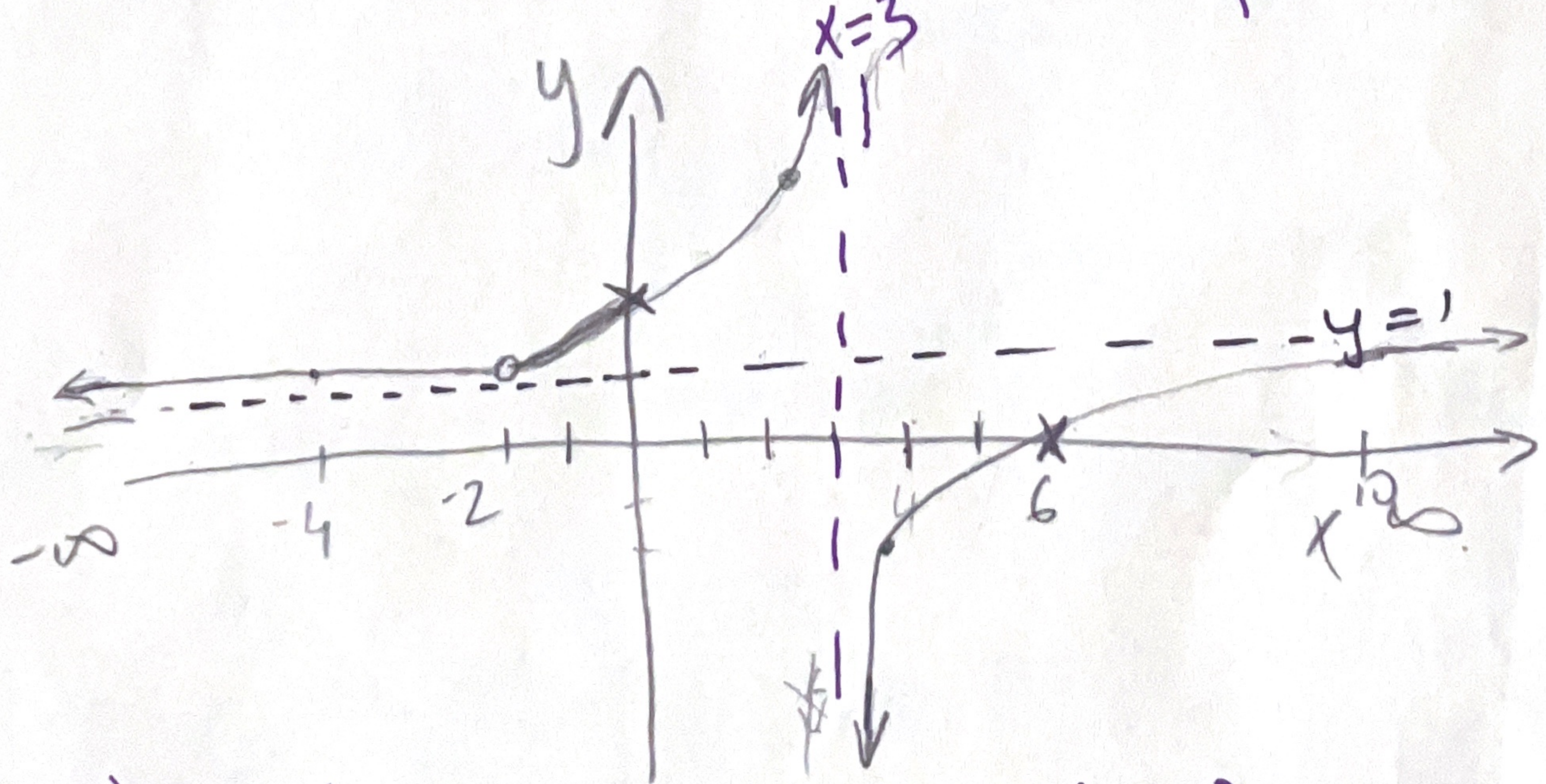
Step 3 HA: $y=1$

y int: $f(0) = \frac{0-6}{0-3} = \frac{-6}{-3} = 2$

Step 4 x int $x-6=0$ $\boxed{x=6}$

Step 5 Graph by pieces

$f(40) = \frac{40-6}{40-3} = \frac{34}{37}$



$f(-4) = \frac{-4-6}{-4-3} = 1.428$

$f(4) = \frac{4-6}{4-3} = \frac{-2}{1} = -2$

$f(10) = \frac{10-6}{10-3} = \frac{4}{7}$