Day 14: *5.2 and 5.3 Polynomials of Higher Degree*

1. Polynomial Functions - Review

In this section we will review polynomial functions and concentrate on how to graph them by hand.

POLYNOMIAL FUNCTIONS

A **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

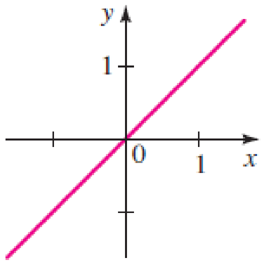
Example:
Answer the following questions about the function $f = -3x^4 + 2x^3 + x + 1$.

1. Is f a polynomial function?
2. What are the coefficients of the polynomial?
3. What is the constant term?
4. What is the leading coefficient?
5. What is the leading term?

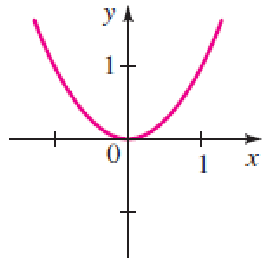
2. Graphs of Polynomial Functions

Graphs of polynomial functions are smooth curves with no breaks or corners.

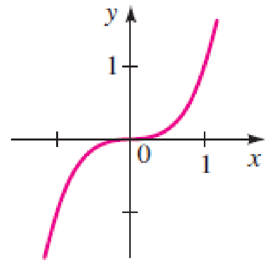
Examples of graphs of monomials (polynomials with single terms):



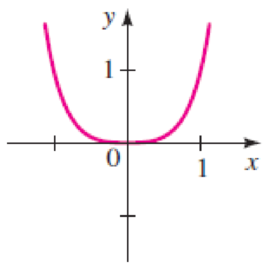
$y = x$



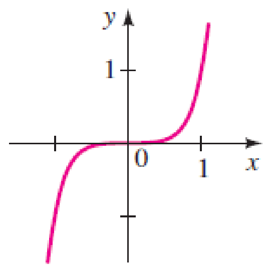
$y = x^2$



$y = x^3$



$y = x^4$



$y = x^5$

The greater the degree of a polynomial, the more complicated its graph can be. The domain of a polynomial function is the set of all real numbers, so we can sketch only a small portion of the graph. In order to draw a sketch of a polynomial function we must take into consideration it intercepts (x intercepts are called zeroes or roots of the polynomial), its end behaviour and its turning points.

3. End behaviour of Polynomial Functions

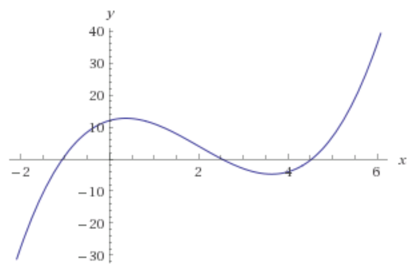
The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction. The end behaviour of a polynomial is given by the leading term. To describe end behavior, we use the following arrow notation.

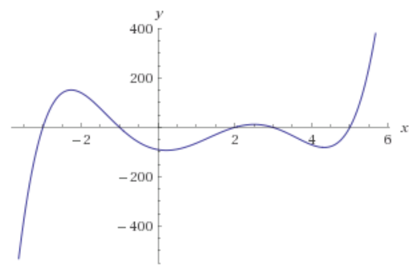
Example: Determine the end behaviour of the polynomial $P_(x) = -2x^4 + 3x + 1$ and $P_2(x) = 5x^5 - 3x^4 + 3x^2 - 10$.

4. Turning points of Polynomial Functions

A turning point occurs whenever the graph of a polynomial function changes from increasing to decreasing or from decreasing to increasing. Turning points are associated with local maxima or local minima on a graph.

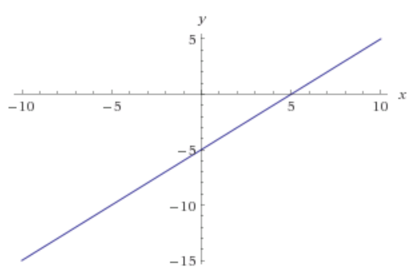
It is possible to use the graph to get a lower estimate of the degree of a polynomial. To do this, we just need to count the number of **turning points** (sometimes called "critical points") of the graph. A turning point is the same thing as a local max or min: it's a point where the graph switches between increasing and decreasing, or vice versa. For example, the polynomials below have 2 and 4 turning points, respectively:

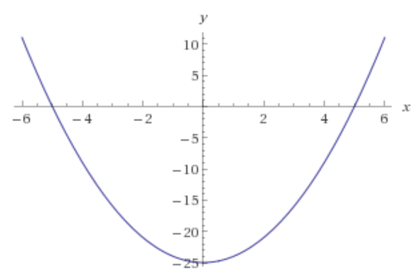




The number of turning points tells us the **smallest possible degree** the polynomial can have. The polynomial on the left must have degree at least $2+1=3$; the one on the right must have degree at least $4+1=5$. In general, if the graph of a polynomial has n turning points, the polynomial's degree is at least $n+1$.

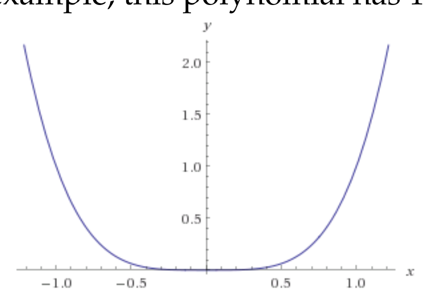
Often, this formula will give the exact degree. For example, these polynomials





have 0 and 1 turning points, so their degrees are at least $0+1=1$ and $1+1=2$. But of course, these are just a linear function and quadratic function, so we know that the degrees are exactly 1 and 2.

It's not always exact, though. For example, this polynomial has 1 turning point



so its degree is at least $1+1=2$. But in fact, its degree is 4, so the estimate was not exact.

5. Roots or zeroes of Polynomial Functions

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property.

For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get $P(x) = (x-2)(x+3)$ From this factored form we easily see that

(a) 2 is a zero of P.

(b) $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.

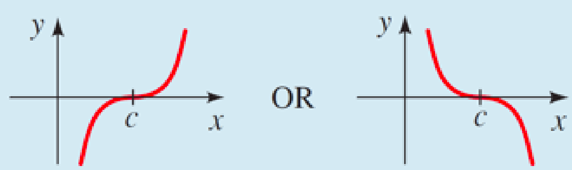
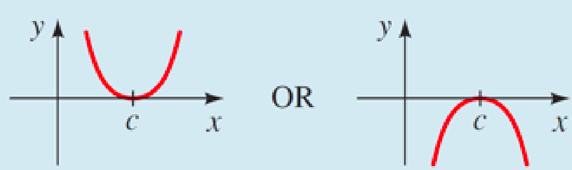
(c) $x - 2$ is a factor of $x^2 + x - 6$.

(d) 2 is an x-intercept of the graph of P.

Multiplicity of zeroes Sometimes when we factor a polynomial there are factors that appear more than once. For example the polynomial $P(x) = (x+1)^2(x-2)(x-1)^3$ has the factor $(x+1)$ appearing twice. We say that its corresponding zero, $x = -1$, has multiplicity 2. Similarly the zero $x = 1$ has multiplicity 3. The shapes of the graphs around zeroes with even or add multiplicity looks different.

SHAPE OF THE GRAPH NEAR A ZERO OF MULTIPLICITY *m*

If c is a zero of P of multiplicity m, then the shape of the graph of P near c is as follows.

Multiplicity of *c*	**Shape of the graph of *P* near the *x*-intercept *c***
m odd, $m > 1$	
m even, $m > 1$	

Practice: Exercise 1

Graph the function $f(x) = (x+2)(x+1)^2(2x-3)$.

Practice: Exercise 2

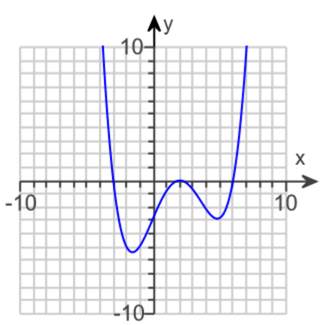
Graph the function $f(x) = -(x+1)^2(x-1)^3(x-2)$.

Practice: Exercise 3

Factor the polynomial and use the factored form to find the real zeroes and then graph it $f(x) = x^3 + 5x^2 - 4x - 20$.

Practice Exercise 4:

Use the graph of the polynomial function f to answer the following.



1. The x-intercepts are:

2. Is the leading coefficient positive or negative?

3. What is the minimum degree of f?

4. Write a formula for $f(x)$.