Mth 146: Day 1
Review: Derivatives, Elementary Integrals & u-Substitution

1 Part 1: Review & Worked Examples

Before we start new material, let's warm up with three things we'll lean on all semester: the core **differentiation rules**, the list of **elementary integrals** (integrals we should just know), and how **u-substitution** lets us extend that list to compositions of functions.

Derivative Rules

Basic Derivatives

$$
\begin{aligned}
(x^n)' &= nx^{n-1} \\
(e^x)' &= e^x \\
(a^x)' &= a^x \ln a \\
(\ln x)' &= \frac{1}{x} \\
(\log_a x)' &= \frac{1}{x \ln a} \\
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\cot x)' &= -\csc^2 x \\
(\sec x)' &= \sec x \tan x \\
(\csc x)' &= -\csc x \cot x \\
(\sin^{-1} x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\cos^{-1} x)' &= \frac{-1}{\sqrt{1-x^2}} \\
(\tan^{-1} x)' &= \frac{1}{1+x^2}
\end{aligned}
$$

Differentiation Rules

- **Constant multiple:** $[cf(x)]' = cf'(x)$
- **Sum/Difference:** $(f \pm g)' = f' \pm g'$
- **Product rule:** $(fg)' = f'g + fg'$
- **Quotient rule:** $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

- **Chain rule:** $[f(g(x))]' = f'(g(x))\, g'(x)$

Example 1. Differentiate $f(x) = x^3 \sin(5x)$.

$f'(x) = (x^3)' \sin(5x) + x^3 (\sin(5x))'$
$= 3x^2 \sin(5x) + x^3 \cdot 5\cos(5x)$

- product rule
$(f \cdot g)' = f'g + fg'$
- chain rule

Example 2. Differentiate $f(x) = \left(\dfrac{x-2}{2x+1}\right)^9$.

$f'(x) = 9\left(\frac{x-2}{2x+1}\right)^{9-1} \cdot \left(\frac{x-2}{2x+1}\right)'$

$= 9\left(\frac{x-2}{2x+1}\right)^8 \cdot \frac{(2x+1)(1) - (x-2)(2)}{(2x+1)^2}$

$= 9 \frac{(x-2)^8}{(2x+1)^8} \frac{2x+1-2x+4}{(2x+1)^2}$

$= \frac{9(x-2)^8 \cdot 5}{(2x+1)^{10}}$

quotient rule
$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

power rule
$(x^n)' = nx^{n-1}$

$(x-2)' = 1$

$(2x+1)' = 2$

Example 3. Differentiate $f(\theta) = e^{\sec(3\theta)}$.

$f'(\theta) = e^{\sec(3\theta)} (\sec 3\theta)'$
$= 3e^{\sec(3\theta)} \cdot \sec(3\theta)\tan(3\theta)$

Example 4. Differentiate $f(x) = \sin(\cos(\tan x))$.

$f'(x) = \cos(\cos(\tan x)) \cdot (\cos(\tan x))'$
$= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot (\tan x)'$
$= -\cos(\cos(\tan x)) \sin(\tan x) \cdot \sec^2 x$

Layer 1: Elementary Integrals

If the integrand already matches one of these forms, we can just write down the antiderivative.

Elementary Integrals

$$
\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1} + C,\;\; n \neq -1 \\
\int \frac{1}{x}\,dx &= \ln|x| + C \\
\int e^x\,dx &= e^x + C \\
\int a^x\,dx &= \frac{a^x}{\ln a} + C \\
\int \sin x\,dx &= -\cos x + C \\
\int \cos x\,dx &= \sin x + C \\
\int \sec^2 x\,dx &= \tan x + C \\
\int \csc^2 x\,dx &= -\cot x + C \\
\int \sec x \tan x\,dx &= \sec x + C \\
\int \csc x \cot x\,dx &= -\csc x + C \\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1} x + C \\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1} x + C \\
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}|x| + C
\end{aligned}
$$

$\sin^{-1} x = \arcsin(x)$

Example 5. $\int_0^3 (x^3 - 6x)\,dx$

$\int x^3\,dx = \frac{x^4}{4} + C$

$= \int_0^3 x^3\,dx - 6\int_0^3 x\,dx$

$= \frac{x^4}{4}\Big|_0^3 - 6\cdot\frac{x^2}{2}\Big|_0^3 = \frac{3^4}{4} - \cancel{6}^{\,3}\cdot\frac{3^2}{\cancel{2}} =$

$= \frac{81}{4} - 27 = -\frac{27}{4}$

Example 6. $\int \left(3e^x - \sec^2 x\right) dx$

$= \int 3e^x dx - \underline{\int \sec^2 x \, dx}$

$= 3e^x - \tan x + C$

Layer 2: Manipulating into Elementary Form

Sometimes the integrand isn't elementary as written, but algebra or a trig identity can rewrite it into pieces that are.

Common Moves

- **Expand** products or powers: $(x+1)^2 = x^2 + 2x + 1$
- **Split** a fraction with one denominator into separate terms: $\dfrac{x^2+3}{x} = x + \frac{3}{x}$
- **Rewrite** radicals and reciprocals as powers of x: $\sqrt{x} = x^{1/2}$, $\dfrac{1}{x^3} = x^{-3}$
- **Use trig identities** to replace a non-elementary trig expression:
  $\tan^2 x = \sec^2 x - 1, \quad \sin^2 x = \dfrac{1 - \cos 2x}{2}$
- **Polynomial long division** when the degree of the numerator is $\geq$ the degree of the denominator: rewrite $\dfrac{p(x)}{q(x)}$ as (quotient) $+\dfrac{\text{remainder}}{q(x)}$

Example 7. $\int \dfrac{x^3 - 4x}{x^2} \, dx$