1 Compound Interest Models

quarterly ⇒ $n=4$
monthly ⇒ $n=12$

1.1 Compound Interest Formula with n Compounding Periods

Definition 1. For an initial principal P, annual interest rate r (expressed as a decimal), time t in years, and n compounding periods per year, the compound interest formula is:

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}} \tag{1}$$

where A is the final amount.

Example 1. If you invest \$5,000 at an annual interest rate of 4% compounded quarterly for 10 years, find the final amount and how long it takes to reach \$9000.

a

b) $P=5{,}000$, $r=4\% = 0.04$, $n=4$

a) Find A if $t=10$ years

$A = P\left(1+\frac{r}{n}\right)^{nt}$

$A = 5000\left(1+\frac{.04}{4}\right)^{(4\cdot 10)}$

$\boxed{A = \$7{,}444.32}$

b) $t=?$ $\quad 9000 = 5000\left(1+\frac{.04}{4}\right)^{4t}$ $\quad / \div 5000$

$\ln\frac{9000}{5000} = \ln\left(1+\frac{.04}{4}\right)^{4t}$

$\dfrac{\ln\left(\frac{9}{5}\right)}{4\ln\left(1+\frac{.04}{4}\right)} = \dfrac{\cancel{4t}\cdot\cancel{\ln\left(1+\frac{.04}{4}\right)}}{\cancel{4\ln\left(1+\frac{.04}{4}\right)}}$

years $14.77 = t$

1.2 Continuously Compounded Interest

Definition 2. As the number of compounding periods approaches infinity, we obtain the continuously compounded interest formula:

$$\boxed{A = Pe^{rt}} \tag{2}$$

where $e \approx 2.71828$ is the base of the natural logarithm.

Example 2. If you invest \$10,000 at an annual interest rate of 5% compounded continuously for 8 years, find the final amount (a) and how long it takes for the amount to double (b).

a) $P = 10{,}000 \quad r = .05, \quad t = 8$ years

$A = Pe^{rt}$

$A = 10{,}000\, e^{(.05)(8)}$

$A = \$14{,}918$

b) $A = Pe^{rt}$

$20{,}000 = 10{,}000\, e^{.05t}$

$\ln 2 = \ln e^{.05t}$

$\ln 2 = .05t \ln e$ $= 1$

$\log_e e = 1$

$\ln 2 = .05t$

$\boxed{\dfrac{\ln 2}{.05} = t}$

$\boxed{t = 13.86 \text{ years}}$

2 Exponential Growth and Decay Models

2.1 Exponential Growth Model

Definition 3. The quantity Q at time t with initial value Q_0 and growth rate k is given by:

$A = Pe^{rt}$

$$Q = Q_0 e^{kt} \quad \text{(continuous)} \tag{3}$$

where $k > 0$ is the continuous growth rate. Q_0

Example 3. A bacterial colony starts with 500 bacteria and grows at a continuous rate of 15% per hour. a) How many bacteria will be present after 12 hours? b) How long it takes for the bacteria to reach 1500?

$r = .15$

a) $Q = Q_0 e^{kt}$

$Q = 500 e^{.15(12)}$

$Q = 3025$ bacteria

b) $Q = Q_0 e^{kt}$

$1500 = 500 \underbrace{e^{.15t}}$

$\frac{1500}{500} = e^{.15t}$

$\ln 3 = \ln e^{.15t}$

$\ln 3 = .15t$

$\frac{\ln 3}{.15} = t$

$t = 7.32$ years

2.2 Exponential Decay Model

Definition 4. The quantity Q at time t with initial value Q_0 and decay rate k is given by:

$$Q = Q_0 e^{-kt} \quad \text{(continuous)} \tag{4}$$

where $k > 0$ is the continuous decay rate.

$Q_0 = 50$

Example 4. A radioactive substance has an initial mass of 50 grams and decays at a continuous rate of 3% per day. a) What will be the mass after 30 days? b) How long until the mass is half of what it started?

$k = .03$

a) $Q = Q_0 e^{-kt} \Rightarrow Q = 50 e^{-.03(30)} = 20.33$ grams

b) $Q = Q_0 e^{-kt}$

$25 = 50 e^{-.03t}$

$\ln \frac{1}{2} = \ln e^{-.03t}$

$\ln(1/2) = -.03t$

$\frac{\ln(1/2)}{-.03} = t$

$t = 23.1$ days

Example 5. A population grows from 10,000 to 15,000 in 5 years. Find the continuous growth rate.

$k = ?$

t_1: $Q = Q_0 e^{kt}$

$10000 = Q_0 e^{kt_1}$ (2)

t_2: $15000 = Q_0 e^{kt_2}$ (1)

$t_2 - t_1 = 5$

$\frac{(1)}{(2)}$

$\frac{15000}{10,000} = \frac{\cancel{Q_0} e^{kt_2}}{\cancel{Q_0} e^{kt_1}}$

$\frac{3}{2} = e^{k\underbrace{(t_2 - t_1)}_{=5}}$

$\ln \frac{3}{2} = \ln e^{k5}$

$\ln(3/2) = 5k$

$\frac{\ln(3/2)}{5} = k$

$k = 8.1\%$