1.2 Finding the Vertex of a Parabola

Definition: Finding the Vertex by Completing the Square

To find the vertex of a quadratic function $f(x) = ax^2 + bx + c$:

1. Factor out the coefficient of x^2 $f(x) = a(x^2 + \frac{b}{a}x) + c$

2. Complete the square for the expression inside the parentheses:

$\frac{b^2}{4a} + f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c$

$f(x) = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$

$(\frac{b}{2a})^2 = \frac{b^2}{4a^2}$

$\cancel{a}\frac{b^2}{4a^{\cancel{2}}} + \frac{b^2}{4a}$

3. Substitute this back into the function:

$f(x) = a(x-h)^2 + k$

$h = -\frac{b}{2a}$ $k = c - \frac{b^2}{4a}$

4. The vertex form is now $f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(c - \frac{b^2}{4a}\right)$

5. The vertex is at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Definition: Vertex Formula

For a quadratic function $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex is:

$$x = -\frac{b}{2a}$$

The y-coordinate can be found by evaluating $f\left(-\frac{b}{2a}\right)$ or by using:

$$y = c - \frac{b^2}{4a}$$

Therefore, the vertex is at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Example 3: Using the Vertex Formula

Find the vertex of the quadratic function $f(x) = -2x^2 + 12x - 13$.

$(h, k) = ?$ $(3, 5)$

$a = -2$

$b = 12$

$c = -13$

$h = -\frac{b}{2a} = -\frac{12}{2(-2)} = 3$ → x coordinate of vertex

$k = f(3) = -2(3)^2 + 12(3) - 13 = -18 + 36 - 13 = 5$

or

$k = \underbrace{c - \frac{b^2}{4a}} = -13 - \frac{(12)^2}{4(-2)} = -13 - \frac{144}{-8} = -13 + 18 = 5$

Example 4: Converting to Vertex Form Using the Vertex Formula

Convert the quadratic function $f(x) = 4x^2 - 16x + 25$ to vertex form using the vertex formula.

$(h, k) = (2, 9)$

$a =$ $b =$ $c =$

$f(x) = a(x-h)^2 + k$

$\boxed{f(x) = 4(x-2)^2 + 9}$

1.3 The Discriminant and Number of Solutions

Definition: The Discriminant

For a quadratic equation $ax^2 + bx + c = 0$, the **discriminant** is defined as:

delta

$$\Delta = b^2 - 4ac$$

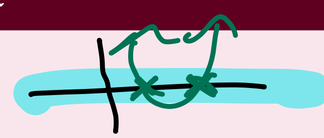
The discriminant determines the number of real solutions to the equation:

Basic Question!!

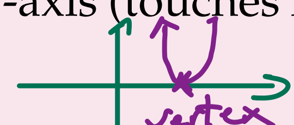
- If $\Delta > 0$, then the equation has two distinct real solutions
- If $\Delta = 0$, then the equation has one real solution (the vertex)
- If $\Delta < 0$, then the equation has no real solution (it has complex solutions)

Note: Graphical Interpretation of the Discriminant

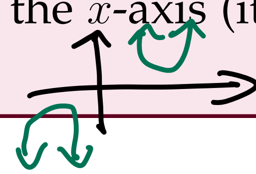
Graphically, for a parabola $y = ax^2 + bx + c$:



- If $\Delta > 0$, the parabola intersects the x-axis at two distinct points.
- If $\Delta = 0$, the parabola is tangent to the x-axis (touches it at exactly one point).



- If $\Delta < 0$, the parabola doesn't intersect the x-axis (it lies entirely above or below the x-axis).



$f(x) = ax^2 + bx + c$

$\pm\sqrt{b}$

quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5: Using the Discriminant

For each of the following quadratic equations, calculate the discriminant and determine the number of real solutions:

a) $2x^2 - 3x + 1 = 0$
b) $4x^2 + 12x + 9 = 0$
c) $3x^2 - 2x + 5 = 0$

a) $\Delta = b^2 - 4ac$
$a = 2,\ b = -3,\ c = 1$
$\Delta = (-3)^2 - 4(2)(1) = 9 - 8 = 1$
$\Delta = 1 > 0$ it has two real solutions
$x = \frac{-(-3) \pm \sqrt{1}}{2 \cdot 2} = \frac{3 \pm 1}{4}$
$\frac{3+1}{4} = 1$
$\frac{3-1}{4} = 1/2$

b) $\Delta = b^2 - 4ac = 12^2 - 4(4)9 = 144 - 144 = 0$
$a = 4,\ b = 12,\ c = 9$
$\Delta = 0$ one real solution : $-\frac{b}{2a} = -\frac{12}{2(4)} = -\frac{3}{2}$

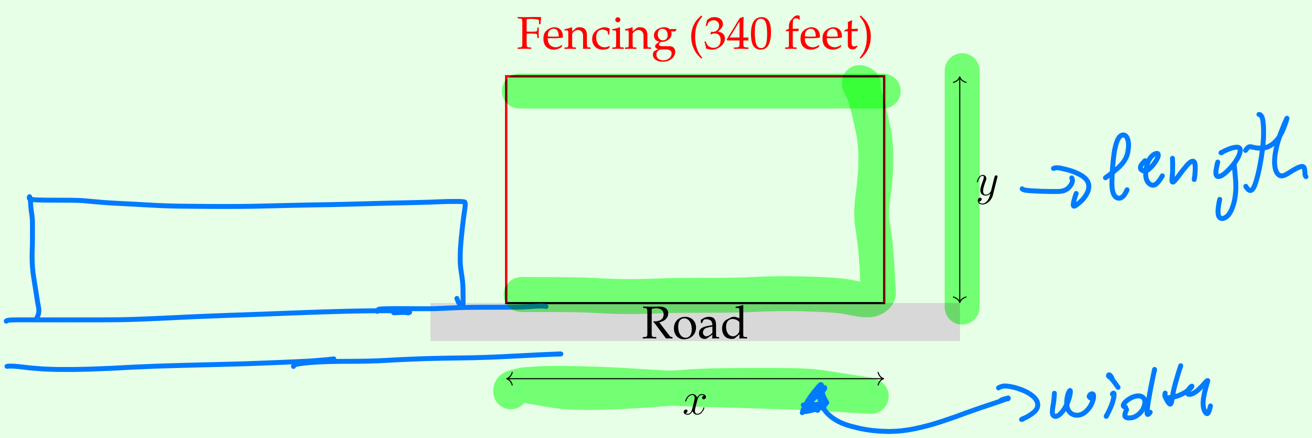
c) $\Delta = b^2 - 4ac = (-2)^2 - 4(3)(5) = 4 - 60 = -56$
$a = 3,\ b = -2,\ c = 5$
$\Delta = -56 < 0$ no real solutions:
Find complex solutions: $x = \frac{2 \pm \sqrt{-56}}{6}$

$56 = 8 \cdot 7$
$56 = 4 \cdot 14$

$x = \frac{-(-2) \pm \sqrt{-56}}{2 \cdot 3} = \frac{2 \pm \sqrt{-1}\sqrt{56}}{6}$
$= \frac{2 \pm i2\sqrt{14}}{6} = \frac{\cancel{2}(1 \pm i\sqrt{14})}{\cancel{6}\,3} = \frac{1 \pm i\sqrt{14}}{3} = \frac{1}{3} \pm \frac{i\sqrt{14}}{3}$

Example 7: Maximizing Area

A parking lot is to be constructed next to a road with 340 feet of fencing available for the remaining three sides. What are the dimensions of the parking lot that would yield the maximum area?



Solution:

Area = (width) (length)

$A = xy$

perimeter (fence)

$x + y + y = 340$

$x + 2y = 340$

solve for y:

$\frac{2y}{2} = \frac{340 - x}{2}$

$y = \frac{340 - x}{2} = 170 - \frac{x}{2}$

$A = x\left(170 - \frac{x}{2}\right)$

$A(x) = x\left(170 - \frac{x}{2}\right) = 170x - \frac{x^2}{2}$

$A(x) = 170x - \frac{x^2}{2}$

$A(x) = -\frac{x^2}{2} + 170x$ (parabola down has a maximum)

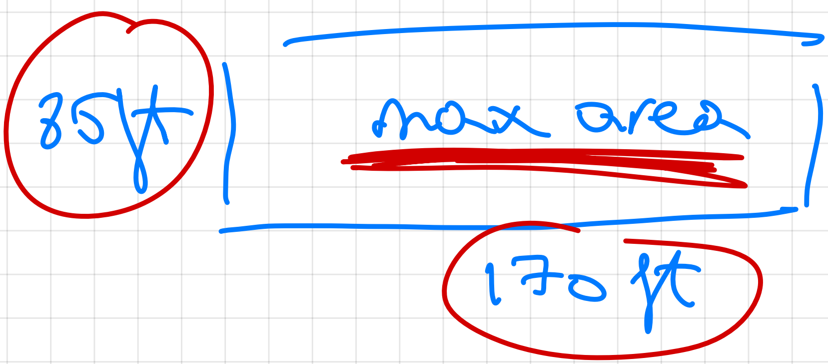
$a = -1/2, \; b = 170, \; c = 0$

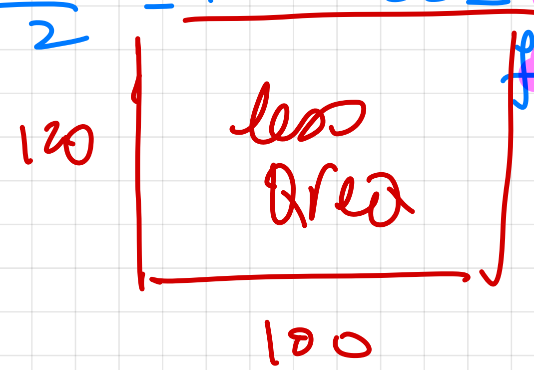
max is given by vertex: (h, k)

$h = -\frac{b}{2a} = \frac{-170}{\cancel{2}\left(-\frac{1}{\cancel{2}}\right)} = \frac{-170}{-1} = 170$

width of 170 ft and length

of $170 - \frac{170}{2} = 170 - 85 = 85$ ft





2 Summary of Key Concepts

Key Concepts Summary

1. A **quadratic function** has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$.
2. The **vertex form** of a quadratic function is $f(x) = a(x-h)^2 + k$ where (h, k) is the vertex.
3. A parabola **opens upward** if $a > 0$ and **opens downward** if $a < 0$.
4. The **axis of symmetry** is the vertical line passing through the vertex: $x = h$ or $x = -\frac{b}{2a}$.
5. If $a > 0$, the function has a **minimum** value at the vertex. If $a < 0$, it has a **maximum** value.
6. The **vertex formula**: For $f(x) = ax^2 + bx + c$, the vertex is at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.
7. The **discriminant** $\Delta = b^2 - 4ac$ determines the number of real solutions:
   - $\Delta > 0$: Two distinct real solutions
   - $\Delta = 0$: One real solution (repeated root)
   - $\Delta < 0$: No real solutions
8. **Completing the square** is a method to convert a quadratic function from general form to vertex form.
9. Applications of quadratic functions include:
   - Projectile motion
   - Maximizing area or revenue
   - Minimizing cost or time