

7.4 Integration of Rational Functions by Partial Fractions

1 Part 1: New Concepts & Worked Examples

The method of Partial Fractions allows us to integrate a rational function $\frac{P(x)}{Q(x)}$, by first expressing it as a sum of simpler fractions called **Partial Fractions** that can be integrated more easily.

For example:

$$\frac{1}{x^2 + 3x - 4} = \frac{1}{5(x - 1)} - \frac{1}{5(x + 4)}$$

The process of breaking down a **proper** rational function into more fractions is called **Partial Fraction Decomposition**.

Recall: A proper rational function

Case 1: The denominator is a product of distinct linear factors

Example 1. $\int \frac{1}{x^2 + 3x - 4} dx$

Case 2: The denominator is a product of linear factors some of which are repeated

Example 2. $\int_{-1}^0 \left(\frac{x^3}{x^2 - 2x + 1} \right) dx.$

Case 3: Denominator contains non-repeating irreducible quadratic factors $ax^2 + bx + c$ where $b^2 - 4ac < 0$

Example 3. $\int \frac{x^3 + 4}{x^2 + 4} dx.$

Example 4. $\int \frac{10}{(x - 1)(x^2 + 9)} dx.$

Case 4: Denominator contains repeating irreducible quadratic factors

Example 5. $\int \frac{dx}{x(x^2 + 4)^2}$

Rationalizing Substitutions

Example 6. $\int_0^1 \frac{1}{1 + \sqrt[3]{x}} dx.$

Example 7. $\int \frac{dx}{x\sqrt{x+1}}.$

2 Part 2: Your Turn — Practice

Try the following individually or in small groups. The final answer is given as a footnote at the bottom of the page — only the final answer, not the steps, so use it to check your work rather than replace it.

1. $\int \left[\frac{2x^3 - 2x^2 + 1}{x^2 - x} \right] dx.$ ¹

¹Answer: $x^2 + \ln \left| \frac{x-1}{x} \right| + C.$

2. $\int \frac{e^x + 1}{e^{2x} - 4} dx.$ ²

Example 8. $\int \frac{12x^3 + 42x^2 - x - 8}{x^4 + 7x^3 - 8x^2} dx.$ ³

²Answer: $-\frac{x}{4} + \frac{3}{8} \ln |e^x - 2| - \frac{1}{8} \ln (e^x + 2) + C.$

³Answer: $\ln |x| - \frac{1}{x} + 6 \ln |x + 8| + 5 \ln |x - 1| + C.$

Example 9. $\int_{-1}^0 \left(\frac{x^3}{x^2 - 2x + 1} \right) dx.$ ⁴

⁴Answer: $2 - 3 \ln 2$.

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Rewrite proper rational functions using the partial fraction method.**

Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

Reflection:

2. **Solve integrals involving rational functions through a combination of long division and partial fraction decomposition.**

Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

Reflection:
