1 Compound Interest Models

1.1 Compound Interest Formula with n Compounding Periods

Definition 1. For an initial principal P, annual interest rate r (expressed as a decimal), time t in years, and n compounding periods per year, the compound interest formula is:

$n=12$, monthly
$n=365$ daily

$$* \quad \boxed{A = P\left(1+\frac{r}{n}\right)^{nt}} \tag{1}$$

where A is the final amount.

Example 1. If you invest \$5,000 at an annual interest rate of 4% compounded quarterly for 10 years, find the final amount and how long it takes to reach \$9000.

$n=4$

a) $A=?$ b) $t=?$

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

a) $A = 5000\left(1+\frac{0.04}{4}\right)^{4\cdot 10}$

calculator: $5000\cdot(1+0.04/4)\wedge(40)$

$A = \$7444.32$

b) $9000 = 5000\left(1+\frac{0.04}{4}\right)^{4t}$ | $\div 5000$

$\frac{9000}{5000} = \left(1+\frac{0.04}{4}\right)^{4t}$ | take log of both sides

$\log\left(\frac{9}{5}\right) = \log\left(1+\frac{.04}{4}\right)^{4t}$

$\log(9/5) = 4\cdot t\cdot \log\left(1+\frac{.04}{4}\right)$ $\Rightarrow$ exact ans $t = \frac{\log(9/5)}{4\log\left(1+\frac{.04}{4}\right)}$

$t = 14.77$ years

1.2 Continuously Compounded Interest

Definition 2. As the number of compounding periods approaches infinity, we obtain the continuously compounded interest formula:

$$\boxed{A = Pe^{rt}} \tag{2}$$

where $e \approx 2.71828$ is the base of the natural logarithm.

Example 2. If you invest \$10,000 at an annual interest rate of 5% compounded continuously for 8 years, find the final amount and how long it takes for the amount to double.

a) $A=?$

b) $t=?$

$A = Pe^{rt}$

a) $A = 10{,}000\, e^{(.05)8} = \$14{,}918$

b) $20{,}000 = 10{,}000\, e^{.05t}$ $\quad |\ \div 10{,}000$

$\frac{20{,}000}{10{,}000} = e^{.05t}$ $\quad |\ \ln$

$\ln 2 = \ln e^{.05t}$

$\ln e = \log_e e = 1$

$\ln 2 = .05t \ln e$

$\frac{\ln 2}{.05} = \frac{.05t}{.05}$

$\boxed{\frac{\ln 2}{.05} = t}$ exact ans

$\boxed{t = 13.86 \text{ years}}$

Formula doubling time for $\boxed{A = Pe^{rt}}$ is $\boxed{\frac{\ln 2}{r}}$

2 Exponential Growth and Decay Models

2.1 Exponential Growth Model

Definition 3. The quantity Q at time t with initial value Q_0 and growth rate k is given by:

$$Q = Q_0 e^{kt} \quad \text{(continuous)} \tag{3}$$

where $k > 0$ is the continuous growth rate.

Example 3. A bacterial colony starts with 500 bacteria and grows at a continuous rate of 15% per hour.
a) How many bacteria will be present after 12 hours?
b) How long it takes for the bacteria to reach 1500?

$$Q = Q_0 e^{kt}$$

a) $Q = 500 e^{(.15)(12)} = 3025$ bacteria

b) $1500 = 500 e^{.15t}$ $/ \div 500$

$\ln \frac{1500}{500} = \ln e^{.15t}$ $/ \ln$

$\ln 3 = .15t$

$t = \frac{\ln 3}{.15} = 7.32$ hours

2.2 Exponential Decay Model

Definition 4. The quantity Q at time t with initial value Q_0 and decay rate k is given by:

$$\boxed{Q = Q_0 e^{-kt}} \quad \text{(continuous)} \tag{4}$$

where $k > 0$ is the continuous decay rate.

Example 4. A radioactive substance has an initial mass of 50 grams and decays at a continuous rate of 3% per day.
a) What will be the mass after 30 days?
b) How long until the mass is half of what it started?

$$\boxed{Q = Q_0 e^{-kt}}$$

a) $Q = 50 e^{-.03(30)} = 20.33$ grams

b) $25 = 50 e^{-.03t}$ $/ \div 50$

$\ln \frac{25}{50} = \ln e^{-.03t} \Longrightarrow \ln(1/2) = -.03t \Rightarrow \boxed{t = \frac{\ln(1/2)}{-.03}}$

Half life $\boxed{t = \frac{\ln(1/2)}{-k}}$ $\boxed{t = 23.10 \text{ days}}$

3 Logistic Growth Model

Definition 5. The logistic growth model describes growth with a carrying capacity. The population P at time t is given by:

$$P(t) = \frac{L}{1 + Ae^{-kt}} \tag{5}$$

where L is the carrying capacity, k is the growth rate, and $\boxed{A = \frac{L-P_0}{P_0}}$ with P_0 being the initial population.

Example 5. A population ~~of deer grows~~ according to a logistic model with a ~~carrying capacity of 10,000 deer~~. If the initial population is 500 deer and the growth rate is $k = 0.2$ per year, find the population after 10 years.

$$P(t) = \frac{L}{1 + Ae^{-kt}}$$

$$A = \frac{L - P_0}{P_0} = \frac{10{,}000 - 500}{500} = 19$$

$$\boxed{P(t) = \frac{10{,}000}{1 + 19e^{-.2(10)}}} = \frac{10{,}000}{\left(1 + 19e^{-.2(10)}\right)}$$

$$P(t) = 2800 \text{ deer}$$

4 Logarithmic Models

Definition 6. Logarithmic models are often used to describe phenomena that grow quickly at first and then level off. A basic logarithmic model has the form:

$$f(x) = a\ln(x) + b \tag{6}$$

where a and b are constants determined by the context.

Example 6. The perceived loudness L (in decibels) of a sound with intensity I relative to a reference intensity I_0 is modeled by:

$I = 1000 I_0$

$$\boxed{L = 10\log_{10}\left(\frac{I}{I_0}\right)} \tag{7}$$

If the intensity of a sound is 1000 times the reference intensity, find its loudness in decibels.

$$L = 10 \log_{10} \frac{1000 \cancel{I_0}}{\cancel{I_0}}$$

$$L = 10\cancel{0} \underbrace{\log_{10} 1000}$$

$$L = 10\cancel{0}(3)$$

$$L = 30\cancel{0} \text{ decibels}$$