Mth 146: Day 1
Review: Derivatives, Elementary Integrals & u-Substitution

1 Part 1: Review & Worked Examples

Before we start new material, let's warm up with three things we'll lean on all semester: the core **differentiation rules**, the list of **elementary integrals** (integrals we should just know), and how u**-substitution** lets us extend that list to compositions of functions.

Derivative Rules

Basic Derivatives

$$
\begin{aligned}
(x^n)' &= nx^{n-1} \\
(e^x)' &= e^x \\
(a^x)' &= a^x \ln a \\
(\ln x)' &= \frac{1}{x} \\
(\log_a x)' &= \frac{1}{x \ln a} \\
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\cot x)' &= -\csc^2 x \\
(\sec x)' &= \sec x \tan x \\
(\csc x)' &= -\csc x \cot x \\
(\sin^{-1} x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\cos^{-1} x)' &= \frac{-1}{\sqrt{1-x^2}} \\
(\tan^{-1} x)' &= \frac{1}{1+x^2}
\end{aligned}
$$

Differentiation Rules

- **Constant multiple:** $[cf(x)]' = cf'(x)$
- **Sum/Difference:** $(f \pm g)' = f' \pm g'$
- **Product rule:** $(fg)' = f'g + fg'$
- **Quotient rule:** $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

- **Chain rule:** $[f(g(x))]' = f'(g(x))\, g'(x)$

Example 1. Differentiate $f(x) = x^3 \sin(5x)$.

Example 2. Differentiate $f(x) = \left(\dfrac{x-2}{2x+1}\right)^9$.

Example 3. Differentiate $f(\theta) = e^{\sec(3\theta)}$.

Example 4. Differentiate $f(x) = \sin(\cos(\tan x))$.

If the integrand already matches one of these forms, we can just write down the antiderivative.

Elementary Integrals

$$\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1} + C,\ n \neq -1\\
\int \frac{1}{x}\,dx &= \ln|x| + C\\
\int e^x\,dx &= e^x + C\\
\int a^x\,dx &= \frac{a^x}{\ln a} + C\\
\int \sin x\,dx &= -\cos x + C\\
\int \cos x\,dx &= \sin x + C\\
\int \sec^2 x\,dx &= \tan x + C\\
\int \csc^2 x\,dx &= -\cot x + C\\
\int \sec x \tan x\,dx &= \sec x + C\\
\int \csc x \cot x\,dx &= -\csc x + C\\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1} x + C\\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1} x + C\\
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}|x| + C
\end{aligned}$$

Example 5. $\int_0^3 (x^3 - 6x)\,dx$

Example 6. $\int \left(3e^x - \sec^2 x\right) dx$

Layer 2: Manipulating into Elementary Form

Sometimes the integrand isn't elementary as written, but algebra or a trig identity can rewrite it into pieces that are.

Common Moves

- **Expand** products or powers: $(x+1)^2 = x^2 + 2x + 1$
- **Split** a fraction with one denominator into separate terms: $\dfrac{x^2+3}{x} = x + \frac{3}{x}$
- **Rewrite** radicals and reciprocals as powers of x: $\sqrt{x} = x^{1/2}$, $\dfrac{1}{x^3} = x^{-3}$
- **Use trig identities** to replace a non-elementary trig expression:
  $\tan^2 x = \sec^2 x - 1, \quad \sin^2 x = \dfrac{1-\cos 2x}{2}$
- **Polynomial long division** when the degree of the numerator is $\geq$ the degree of the denominator: rewrite $\dfrac{p(x)}{q(x)}$ as (quotient) $+\dfrac{\text{remainder}}{q(x)}$

Example 7. $\int \dfrac{x^3 - 4x}{x^2}\, dx$

Example 8. $\int \sqrt{x}\,(x-3)\,dx$

Example 9. $\int \tan^2 x\,dx$

Layer 3: u-Substitution

If the integrand is a composition $f(g(x))$ and the derivative of the "inside function" $g(x)$ also shows up (up to a constant multiple), substitution turns it into an elementary integral.

u-Substitution Formula:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du, \qquad u = g(x)$$

Steps:

1. Choose $u = g(x)$ — the inside piece: something raised to a power, inside a root/exponential/log, or in a denominator.
2. Compute $du = g'(x)\,dx$ and solve for the piece you need.
3. Rewrite the *entire* integral in terms of u only.
4. Integrate with respect to u, then substitute $g(x)$ back in for u.

Example 10. $\int 2x\cos(x^2)\,dx$

Example 11. $\displaystyle\int \frac{x}{x^2+1}\,dx$

Example 12. $\displaystyle\int \frac{x^3+3x+1}{x^2+1}\,dx$

Example 13. $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$

Sometimes after substituting, a leftover x still remains in the integrand. If so, solve $u = g(x)$ for x and substitute that expression in as well before integrating.

Example 14. $\int (1+x)\sqrt{2-x}\, dx$

2 Part 2: Your Turn — Practice

Try the following individually or in small groups. Final answers are given as footnotes[1] at the bottom of the page — only the final answer, not the steps, so use them to check your work rather than replace it.

1. Differentiate $f(x) = \dfrac{\ln x}{x^2}$.[2]

2. $\displaystyle\int \left(x^3 + \frac{1}{x} - \sin x\right) dx$.[3]

3. $\displaystyle\int \frac{x^2+1}{\sqrt{x}}\, dx$.[4]

[1] Footnotes are used here (rather than a printed answer key) because a properly tagged footnote stays linked to its problem for screen-reader users, instead of living disconnected on a separate answer page.

[2] Answer: $f'(x) = \dfrac{1 - 2\ln x}{x^3}$.

[3] Answer: $\dfrac{x^4}{4} + \ln|x| + \cos x + C$.

[4] Answer: $\dfrac{2}{5}x^{5/2} + 2x^{1/2} + C$.

4. $\displaystyle\int \cot^2 x\,dx.$[5]

5. $\displaystyle\int 4x^3\sqrt{x^4+5}\,dx.$[6]

6. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx.$[7]

[5] Answer: $-\cot x - x + C \quad (\text{using } \cot^2 x = \csc^2 x - 1).$

[6] Answer: $\dfrac{2}{3}(x^4+5)^{3/2} + C.$

[7] Answer: $\dfrac{1}{2}\ln\bigl(1+e^{2x}\bigr) + C.$

3 Part 3: Reflection

For each learning outcome below, rate how comfortable you feel right now, and write a few sentences: What clicked? What's still fuzzy? What would help?

1. **Recall and apply the basic differentiation rules (power, product, quotient, chain).**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:

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2. **Recognize and integrate elementary functions directly from the list.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:

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3. **Use algebra or a trig identity to rewrite an integrand into elementary form.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection:

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4. **Apply u-substitution to integrate a composition of functions.**

   Comfort level (circle one): 1 — not comfortable 2 3 4 5 — very comfortable

   Reflection: