Exam 3 Study Guide

Polynomials, Rational, Exponential, Logarithmic Functions, and Conic Sections

1 Polynomial Functions

1.1 Basic Concepts

- A polynomial function has the form: $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$
- n is a nonnegative integer (degree of the polynomial)
- $a_n \neq 0$ (leading coefficient)
- Terms:
  - Coefficients: $a_0, a_1, a_2, ..., a_n$
  - Constant term: a_0
  - Leading coefficient: a_n
  - Leading term: $a_n x^n$

1.2 Graphs of Polynomial Functions

- Polynomial graphs are **smooth curves with no breaks or corners**
- The domain of a polynomial function is the set of all real numbers
- Important features to identify:
  - Intercepts (x-intercepts are called zeros or roots)
  - End behavior
  - Turning points

1.3 End Behavior of Polynomial Functions

- End behavior describes what happens as x becomes very large in the positive or negative direction
- End behavior is determined by the leading term of the polynomial
- For $f(x) = a_n x^n$:
  - If n is even and $a_n > 0$: $y \to \infty$ as $x \to \pm\infty$
  - If n is even and $a_n < 0$: $y \to -\infty$ as $x \to \pm\infty$
  - If n is odd and $a_n > 0$: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$
  - If n is odd and $a_n < 0$: $y \to -\infty$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$

1.4 Turning Points of Polynomial Functions

- A turning point occurs when the graph changes from increasing to decreasing or vice versa
- Turning points correspond to local maxima and minima
- If a polynomial has n turning points, its degree is at least $n + 1$

1.5 Roots or Zeros of Polynomial Functions

- To find zeros, factor and use the Zero-Product Property
- Multiplicity of zeros:
  - If $(x - c)^m$ is a factor, then c has multiplicity m
  - Affects the behavior of the graph at the x-intercept:
    * When m is odd ($m > 1$): graph crosses the x-axis at c
    * When m is even ($m > 1$): graph touches the x-axis at c but does not cross it

2 Real Zeros of Polynomial Functions

2.1 The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$, then there exist unique polynomials $q(x)$ (quotient) and $r(x)$ (remainder) such that:

$$f(x) = d(x)q(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.

Alternative form: $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$

2.2 The Remainder Theorem

If a polynomial $f(x)$ is divided by $(x - a)$, the remainder equals $f(a)$.

2.3 The Factor Theorem

A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

2.4 The Rational Zero Test

For a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ with integer coefficients:

- Every rational zero has the form $\frac{p}{q}$ (in lowest terms)
- p is a factor of the constant term a_0
- q is a factor of the leading coefficient a_n

2.5 The Fundamental Theorem of Algebra

- Every polynomial function with degree $n \geq 1$ can be factored into n linear factors
- Form: $f(x) = a(x - r_1)(x - r_2)...(x - r_n)$
- Where $r_1, r_2, ..., r_n$ are complex numbers and $a \neq 0$

2.6 Complex Zeros

- If a polynomial has real coefficients and a complex zero $a + bi$ (where $b \neq 0$), then its complex conjugate $a - bi$ is also a zero
- Complex zeros come in conjugate pairs

3 Rational Functions

3.1 Definition

A rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$ where:

- $P(x)$ and $Q(x)$ are polynomials
- $Q(x) \neq 0$

3.2 Domain

The domain of a rational function is all real numbers except those for which the denominator equals zero.

3.3 Holes in Rational Functions

- If both the numerator and denominator have a common factor $(x - a)$ that reduces, then the rational function has a hole at $x = a$
- To find the y-coordinate of the hole, evaluate the reduced form at $x = a$

3.4 Vertical Asymptotes

A rational function has vertical asymptotes at values of x where:

- The denominator equals zero
- The numerator does not equal zero at that value (no common factor)

3.5 Horizontal Asymptotes

For a rational function $f(x) = \frac{P(x)}{Q(x)}$:

- If degree of $P(x) <$ degree of $Q(x)$: Horizontal asymptote at $y = 0$
- If degree of $P(x) =$ degree of $Q(x)$: Horizontal asymptote at $y = \frac{a}{b}$, where a and b are the leading coefficients of $P(x)$ and $Q(x)$, respectively
- If degree of $P(x) >$ degree of $Q(x)$: No horizontal asymptote

4 Exponential Functions

4.1 Definition

An exponential function is a function of the form $f(x) = b^x$ where:

- The base $b > 0$ and $b \neq 1$
- The independent variable x is in the exponent

4.2 Natural Exponential Function

The natural exponential function has the form $f(x) = e^x$, where $e \approx 2.71828$

4.3 Properties of Exponential Functions

- Domain: All real numbers
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$ since $b^0 = 1$
- No x-intercepts
- Horizontal asymptote: $y = 0$ as $x \to -\infty$
- Always increasing if $b > 1$
- Always decreasing if $0 < b < 1$

4.4 Transformations of Exponential Functions

Apply standard transformation rules to exponential functions:

- Vertical shift: $f(x) = b^x + k$
- Horizontal shift: $f(x) = b^{x-h}$
- Reflection: $f(x) = -b^x$ or $f(x) = b^{-x}$
- Vertical stretch/compression: $f(x) = a \cdot b^x$

5 Logarithmic Functions

5.1 Definition

The logarithmic function with base b is the inverse of the exponential function with base b. If $y = b^x$, then $x = \log_b(y)$

5.2 Special Logarithms

- Common logarithm: $\log_{10}(x)$ or simply $\log(x)$
- Natural logarithm: $\log_e(x)$ or $\ln(x)$

5.3 Properties of Logarithmic Functions

- Domain: $(0, \infty)$
- Range: All real numbers
- x-intercept: $(1, 0)$ since $\log_b(1) = 0$
- Vertical asymptote: $x = 0$
- Increasing if $b > 1$
- Decreasing if $0 < b < 1$

5.4 Basic Logarithmic Properties

- $\log_b(1) = 0$
- $\log_b(b) = 1$
- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$
- $\log_b(xy) = \log_b(x) + \log_b(y)$ (product property)
- $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$ (quotient property)
- $\log_b(x^p) = p \cdot \log_b(x)$ (power property)

5.5 Converting Between Exponential and Logarithmic Forms

- If $\log_2(8) = 3$, then $2^3 = 8$
- If $\log_{10}(7) = x$, then $10^x = 7$

5.6 Domain of Logarithmic Expressions

For logarithmic expressions, determine the domain by finding values of x for which the input expression is positive.

6 Exponential and Logarithmic Equations and Models

6.1 Compound Interest Models

- Compound interest formula with n compounding periods: $A = P(1 + \frac{r}{n})^{nt}$ where:
  - A = final amount
  - P = principal (initial investment)
  - r = annual interest rate (as a decimal)
  - n = number of compounding periods per year
  - t = time in years
- Continuously compounded interest: $A = Pe^{rt}$

6.2 Exponential Growth and Decay Models

- Exponential growth model: $Q = Q_0 e^{kt}$ where:
  - Q = quantity at time t
  - Q_0 = initial quantity
  - $k > 0$ = continuous growth rate
  - t = time
- Exponential decay model: $Q = Q_0 e^{-kt}$ where $k > 0$ is the continuous decay rate

6.3 Finding Growth/Decay Rates

To find the continuous growth rate when given two data points:

1. Set up the equation with two data points: $Q_2 = Q_0 e^{kt_2}$ and $Q_1 = Q_0 e^{kt_1}$
2. Divide the equations to eliminate Q_0: $\frac{Q_2}{Q_1} = e^{k(t_2 - t_1)}$
3. Take the natural logarithm of both sides: $\ln(\frac{Q_2}{Q_1}) = k(t_2 - t_1)$
4. Solve for k: $k = \frac{\ln(\frac{Q_2}{Q_1})}{t_2 - t_1}$

7 Solving Exponential and Logarithmic Equations

7.1 Solving Exponential Equations

Steps:

1. Isolate the exponential expression
2. Take the logarithm of both sides
3. Simplify using logarithmic properties
4. Solve for the variable

7.2 Solving Logarithmic Equations (with Same Base)

Steps:

1. Condense logs into a single log if multiple logs are on one side
2. Solve the equation using the arguments
3. Check and discard extraneous solutions

7.3 Solving Logarithmic Equations (with Terms Without Logs)

Steps:

1. Isolate terms with a log of the same base to one side
2. Condense logs into a single log and isolate
3. Write as an exponential equation
4. Solve the equation
5. Check and discard extraneous solutions

8 Circles and Ellipses (Day 24)

8.1 Circles

- Definition: A circle is the set of all points in a plane that are equidistant from a fixed point called the center. The fixed distance from any point on the circle to the center is called the radius.
- Standard form of the equation of a circle:

  $$(x-h)^2 + (y-k)^2 = r^2$$

  where (h, k) is the center and r is the radius.

- Special case: Circle centered at the origin $(0, 0)$:

$$x^2 + y^2 = r^2$$

- General form of a circle's equation:

$$x^2 + y^2 + Ax + By + C = 0$$

- Converting general form to standard form (completing the square):
  1. Group the x terms and y terms separately
  2. Complete the square for each group
  3. Rewrite in standard form

8.2 Distance and Midpoint Formulas

- Distance Formula: The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Midpoint Formula: The midpoint $M(x, y)$ of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

8.3 Ellipses

- Definition: An ellipse is the set of all points in a plane for which the sum of the distances from two fixed points (foci) is constant.
- Standard form of the equation of an ellipse centered at the origin:
  - Horizontal major axis: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b > 0$)
  - Vertical major axis: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (where $a > b > 0$)
- Key points of an ellipse with horizontal major axis:
  - Vertices: $(\pm a, 0)$
  - Co-vertices: $(0, \pm b)$
  - Foci: $(\pm c, 0)$ where $c^2 = a^2 - b^2$
- Key points of an ellipse with vertical major axis:
  - Vertices: $(0, \pm a)$
  - Co-vertices: $(\pm b, 0)$

  - Foci: $(0, \pm c)$ where $c^2 = a^2 - b^2$

- Standard form of the equation of an ellipse centered at (h, k):
  - Horizontal major axis: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  - Vertical major axis: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$